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A robust WG finite element method for convection–diffusion–reaction equations

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ABSTRACT

This paper proposes and analyzes a weak Galerkin (WG) finite element method for 2- and 3-dimensional convection–diffusion–reaction problems on conforming or nonconforming polygon/polyhedral meshes. The WG method uses piecewise-polynomial approximations of degrees k ($k \geq 0$) for both the scalar function and its trace on the inter-element boundaries. We show that the method is robust in the sense that the derived a priori error estimates is uniform with respect to the coefficients for sufficient smooth true solutions. Numerical experiments confirm the theoretical results.

1. Introduction

In this paper, we consider a weak Galerkin (WG) finite element method for the following convection–diffusion–reaction equation: seek a scalar function u such that

$$\begin{cases} -\varepsilon\Delta u + \nabla\cdot(\boldsymbol{b}u) + \sigma u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a polygonal or polyhedral domain, $f \in L^2(\Omega)$, $\varepsilon > 0$, $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}) \in [W^{1,\infty}(\Omega)]^d$, $\sigma = \sigma(\boldsymbol{x}) \in L^\infty(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$.

Similarly to [1], for the subsequent stability analysis we introduce several assumptions, **A1**–**A3**, on the velocity vector $\boldsymbol{b}$ and the "effective" reaction function $\bar\sigma := \sigma + \frac{1}{2}\nabla\cdot\boldsymbol{b}$.

A1. $\bar\sigma$ has a nonnegative lower bound, i.e.,

$$\sigma_0 := \inf_{\boldsymbol{x}\in\Omega}\bar\sigma \geq 0. \tag{1.2}$$

A2. $\boldsymbol{b}$ has no closed curves and

$$|b(\boldsymbol{x})| \neq 0 \quad \text{for all } x \in \Omega.$$

A3. There exist two positive constants, c_b and c_σ, such that

$$\|\boldsymbol{b}\|_{0,\infty} \geq c_b\|\boldsymbol{b}\|_{1,\infty}, \tag{1.3}$$

$$\sigma_0 + b_0 \geq c_\sigma\|\bar\sigma\|_{0,\infty} \quad \text{with } b_0 := \|\boldsymbol{b}\|_{0,\infty}/L, \tag{1.4}$$

where L is the diameter of Ω.

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As shown in [1], the assumption **A2** implies that there exists a function $\psi \in W^{k+1,\infty}(\Omega)$ for any integer $k \geq 0$, such that

$$\boldsymbol{b} \cdot \nabla\psi \geq 2b_0 > 0, \quad \forall \boldsymbol{x} \in \Omega. \tag{1.5}$$

It is well-known that standard finite element methods often suffer from the deterioration of numerical accuracy for convection-dominated convection–diffusion equations due to local singularities arising from interior or boundary layers. A lot of research has been devoted to solving such kinds of problems properly, such as stabilized methods [2–6] and discontinuous Galerkin (DG) methods [1,7,4,5,8–10].

In most DG methods, numerical fluxes are specified at the interfaces of elements, and the convection field $\boldsymbol{b}$ is assumed to be either constant or divergence-free. By following a different way, a class of DG methods were analyzed in [1] for the convection–diffusion–reaction model (1.1), where the weighted-residual approach of [11] was applied to derive the DG formulations. The methods are shown to be well suited for not only convection-dominated regimes, but also diffusion-dominated, reaction-dominated and intermediate regimes. However, an inconvenient feature of the DG methods is that the penalization parameter for stability is required to be a "sufficiently" large (practically unknown) number. This was remedied by local DG (LDG) methods [4] and hybridizable DG (HDG) methods [12–14], where the diffusion coefficient $\boldsymbol{a}$ is a symmetric $d \times d$ matrix function that is uniformly positive definite on Ω and thus the analyses therein exclude the convection-dominated case. In [15], a class of HDG methods were applied to the convection–diffusion problems and shown to be well suited for convection-dominated regimes.

Closely related to the HDG framework is the weak Galerkin (WG) method developed in [16,17] for second-order elliptic problems. The WG method is designed by using a weakly defined gradient operator over functions with discontinuity, and allows the use of totally discontinuous functions in the finite element procedure. Similar to the HDG framework, the WG method allows the use of conforming or nonconforming meshes of arbitrary polygons/polyhedrals, and possesses the property of local elimination, i.e., the unknowns defined in the interior of elements can be eliminated locally by using the unknowns defined on the interfaces of all elements. Applications of WG methods to different types of PDEs can be found in [18,19,17,20–23]. We refer to [24] for a modified WG Galerkin finite element method for convection–diffusion problems in 2D.

The goal of this paper is to propose and analyze a class of robust WG finite element methods for the convection–diffusion–reaction problem (1.1). We note that the convection field $\boldsymbol{b}$ is not assumed to be divergence-free and the reaction coefficient σ is a function. We will show that the proposed methods are of convergence rates independent of the coefficients ε, $\boldsymbol{b}$ and σ, providing sufficiently smooth solutions. We make a simple comparison in Table 1.1 between the methods of [1,15] and ours. We mention that, due to the property of local elimination, the WG and HDG methods lead to discrete systems of smaller sizes than the same order DG methods. It should be pointed out that the elliptic problem considered in [16] is actually in the convection–diffusion–reaction format. However, the WG scheme therein was analyzed only for simplex meshes and the derived error estimates are not uniform with respect to the coefficients.

The rest of this paper is organized as follows. In Section 2, we introduce the WG scheme. Section 3 derives stability and Section 4 is devoted to the error estimation. Finally, Section 5 provides several numerical examples to verify our theoretical results.

Throughout this paper, we use C to denote a positive constant independent of h, h_T, h_E, $\boldsymbol{b}$, ε and σ, and not necessarily the same at its each occurrence. For simplicity we use $a \lesssim b$ ($a \gtrsim b$) to represent $a \leq Cb$ ($a \geq Cb$).

2. WG finite element method

2.1. Notations and preliminary results

For any bounded domain $\Lambda \subset \mathbb{R}^s$ ($s = d, d-1$), let $H^m(\Lambda)$ and $H_0^m(\Lambda)$ denote the usual mth-order Sobolev spaces on Λ, and $\|\cdot\|_{m,\Lambda}$, $|\cdot|_{m,\Lambda}$ denote the norm and semi-norm on these spaces. We use $(\cdot,\cdot)_{m,\Lambda}$ to denote the inner product of $H^m(\Lambda)$, with $(\cdot,\cdot)_\Lambda := (\cdot,\cdot)_{0,\Lambda}$. When $\Lambda = \Omega$, we denote $\|\cdot\|_m := \|\cdot\|_{m,\Omega}$, $|\cdot|_m := |\cdot|_{m,\Omega}$, $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$. In particular, when $\Lambda \in \mathbb{R}^{d-1}$, we use $\langle\cdot,\cdot\rangle_\Lambda$ to replace $(\cdot,\cdot)_\Lambda$. We note that bold face fonts will be used for vector (or tensor) analogues of the Sobolev spaces along with vector-valued (or tensor-valued) functions. For an integer $k \geq 0$, $P_k(\Lambda)$ denotes the set of all polynomials defined on Λ with degree not greater than k.

Let $\mathcal{T}_h = \bigcup\{T\}$ be a shape regular partition (to be defined later) of the domain Ω consists of arbitrary polygons. We note that $\mathcal{T}_h$ can be a conforming partition or a nonconforming partition which allows hanging nodes.

For any $T \in \mathcal{T}_h$, we let h_T be the infimum of the diameters of circles (or spheres) containing T and denote the mesh size $h := \max_{T\in\mathcal{T}_h} h_T$.

An edge (or face) E on the boundary ∂T of T is called a proper edge (or face) if the endpoints (or vertexes) of the edge (or face) E are the nodes of $\mathcal{T}_h$ and no other nodes of $\mathcal{T}_h$ are on E. See Fig. 2.1 for example, EF, FH and HI are proper edges, while EH, FI and EI are not. Let $\mathcal{E}_h = \bigcup\{E\}$ be the union of all proper edges (faces) of $T \in \mathcal{T}_h$. We denote by h_E the length of edge E if $d = 2$ and the infimum of the diameters of circles containing face E if $d = 3$. For all $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$, we denote by $\boldsymbol{n}_T$ and $\boldsymbol{n}_E$ the unit outward normal vectors along ∂T and E, respectively.

The partition $\mathcal{T}_h$ is called shape regular in the sense that assumptions **M1**–**M2** hold true.