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Krylov subspace-based model reduction for a class of bilinear descriptor systems[☆]

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ABSTRACT

We consider model order reduction for bilinear descriptor systems using an interpolatory projection framework. Such nonlinear descriptor systems can be represented by a series of generalized linear descriptor systems (also called subsystems) by utilizing the Volterra–Wiener approach (Rugh, 1981). Standard projection techniques for bilinear systems utilize the generalized transfer functions of these subsystems to construct an interpolating approximation. However, the resulting reduced-order system may not match the polynomial parts of the generalized transfer functions. This may result in an unbounded error in terms of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms. In this paper, we derive an explicit expression for the polynomial part of each subsystem by assuming a special structure of the bilinear system which reduces to an index-1 linear descriptor system or differential algebraic equation (DAE) if the bilinear terms are zero. This allows us to propose an interpolatory technique for bilinear DAEs which not only achieves interpolation, but also retains the polynomial parts of the bilinear systems. The approach extends the interpolatory technique for index-1 linear DAEs (Beattie and Gugercin, 2009) to bilinear DAEs. Numerical examples are used to illustrate the theoretical results.

1. Introduction

The importance of model order reduction arises in the analysis of high order mathematical models that describe complex dynamical systems. These high order models are often expensive to analyze and therefore, they are replaced by reduced-order systems to simulate the approximate behavior of the actual system. Various approaches have been developed for model order reduction, see, e.g., [1–4]. In case of linear systems, balanced truncation [5], moment-matching methods [6] and the iterative rational Krylov method [7] are well-used and well-established model reduction methods. However, most practical systems have nonlinearities and model reduction of such systems, particularly models described by differential algebraic equations (DAEs), also called descriptor systems, are less developed and require further research.

In this paper, we investigate Krylov projection methods for bilinear descriptor systems. In general, a bilinear descriptor system has the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + \sum_{i=1}^{m} N^{(i)}x(t)u_i(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

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where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. The matrices $E, A, N^{(i)}, i = 1, \ldots, m, B, C$ and D are all real with dimensions determined by those of $x(t)$, $u(t)$ and $y(t)$. Notice that the bilinear terms in the system, involving the product of states and inputs, make it a special class of nonlinear systems. Also, the matrix E might be singular, but it is assumed that the matrix pencil $\alpha E - \beta A$ is regular, that is

$$\det(\alpha E - \beta A) \neq 0, \quad \text{for some } (\alpha, \beta) \in \mathbb{C}^2.$$

The generalized eigenvalues of the matrix pencil $\lambda E - A$ are defined by pairs $(\alpha_i, \beta_i) \in \mathbb{C}^2 \backslash \{0, 0\}$ such that $\det(\alpha_i E - \beta_i A) = 0$. The pairs corresponding to $\beta_i \neq 0$ are the finite eigenvalues of the matrix pencil, given as $\lambda_i = \alpha_i/\beta_i$, and on the other hand, the pairs corresponding to $\beta_i = 0$, are called infinite eigenvalues of the matrix pencil. In this paper, we also assume that the matrix pencil $\lambda E - A$ is *c-stable*, that is all the finite eigenvalues of the matrix pencil lie in the open left half plane. These assumptions are made in order to ensure the existence and uniqueness of smooth solutions to the dynamical system for sufficiently smooth inputs. For more details, we refer to [8].

Moreover, if the matrix pencil $\lambda E - A$ is regular, then there exist nonsingular matrices X and Y, transforming the pencil into the Weierstrass canonical form [9,10]:

$$E = X \begin{bmatrix} I & 0 \\ 0 & \mathcal{N} \end{bmatrix} Y, \qquad A = X \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix} Y,$$

where the Jordan matrix J is such that its eigenvalues coincide with the finite eigenvalues of the matrix pencil, and $\mathcal{N}$ is a nilpotent matrix corresponding to the infinite eigenvalues. If the index of nilpotency of $\mathcal{N}$ is $\nu > 0$, then $\mathcal{N}^\nu = 0$ and $\mathcal{N}^{\nu-1} \neq 0$. This nilpotency index is often called the *index* of the matrix pencil $\lambda E - A$.

For the nonlinear descriptor system (1), the problem of model order reduction is to derive another system with much smaller state-space dimension $r \ll n$, similar to (1), i.e.,

$$\begin{aligned} E_r \dot{x}_r(t) &= A_r x_r(t) + \sum_{i=1}^{m} N_r^{(i)} x_r(t) u_i(t) + B_r u(t), \\ y_r(t) &= C_r x_r(t) + D_r u(t) \end{aligned} \tag{2}$$

such that the output behavior and some important properties of (1) are retained by (2) for an admissible set of input functions $u(t)$. The reduced-order system (2) can be obtained via projections as follows:

- Construct basis matrices $V \in \mathbb{R}^{n\times r}$ and $W \in \mathbb{R}^{n\times r}$ for the subspaces $\mathcal{V}$ and $\mathcal{W}$ respectively.
- Approximate $x(t)$ by $Vx_r(t)$.
- Ensure the Petrov–Galerkin condition:

$$W^T \left(EV\dot{x}_r(t) - AVx_r(t) - \sum_{i=1}^{m} N^{(i)} V x_r(t) u_i(t) - Bu(t) \right) = 0.$$

As a result, the state matrices associated with the reduced-order system (2) are given by

$$E_r = W^T EV, \qquad A_r = W^T AV, \qquad N_r^{(i)} = W^T N^{(i)} V,$$
$$B_r = W^T B, \qquad C_r = CV.$$

Clearly, for a given system, the reduced-order system obtained via projection depends on the choice of V and W, or equivalently, on the subspaces $\mathcal{V}$ and $\mathcal{W}$. If the matrix E is the identity matrix or nonsingular, these basis matrices and the resulting reduced-order system can be computed by extending the standard balanced truncation and interpolatory projection methods from linear to bilinear systems [11–18]. The bilinear version of balanced truncation involves the solutions of two generalized Lyapunov equations, which are known to be computationally complex [14]. However, in [19,20] effective methods for solving these Lyapunov equations are suggested. Its extension to the descriptor case also is an open problem, though. However, in this work, we focus on interpolatory projection methods for descriptor systems.

Recently, for linear descriptor systems it was shown [21] that it is necessary for interpolatory techniques to compute a reduced-order system which not only interpolates the actual transfer function of the system, but also retains its polynomial part in order to ensure a bounded error in terms of the $\mathcal{H}_2$-norm. We extend this observation to bilinear descriptor systems. The idea is to compute a reduced-order system for a given bilinear DAE system such that the generalized transfer functions associated with the reduced-order and the actual bilinear systems not only interpolate at some predefined interpolation points, but also match their corresponding polynomial parts. This involves, first, identifying the generalized transfer functions of the bilinear DAE system which is possible by using the Volterra series representation [22]. Secondly, we construct the basis matrices V and W, where the first k generalized transfer functions are used, similar to the standard interpolatory subspaces [15]. Subsequently, we identify the polynomial part of each generalized transfer function and finally project the bilinear DAE system to obtain the required reduced-order system.

It is not straightforward to identify explicitly the polynomial parts of the generalized transfer functions. In this paper, we assume a special structure of bilinear systems which allows us to compute explicitly a constant polynomial part of each generalized transfer function. The special structure reduces to an index-1 linear DAE system, if the bilinear terms are zero.