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Letter to the editor

Lower bound limit analysis by quadrilateral elements

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ABSTRACT

This paper presents a quadrilateral element formulation of lower bound theorem. The formulation uses a four-noded quadrilateral element. The weak form of the equilibrium equations is performed to linearize the equilibrium equations. By Green's theorem, the integral over quadrilateral element are transformed into boundary integral over the element boundary. The major advantage of using quadrilateral element, rather than triangular element, is that more accurate lower bound can be obtained with the same element size.

Two numerical examples are given to illustrate the capability of the new method for computing lower bound. The accuracy of the quadrilateral element formulation is compared with that of three-noded triangular element formulation in detail.

1. Introduction

As an effective method to estimate the ultimate bearing capacity for structures, lower- and upper-bound limit analysis have been extensively used in the past in a wide variety of problems, such as tunnels [1,2], slopes [3,4], foundations [5,6], anchors [7,8], braced excavations [9,10]. Accurate assessment of the stability of those structures is an important task.

To make the 'gap' between the upper- and lower-bound limit loads smaller, some adaptive meshing strategies have been developed, for example, based on the deformations and on the slack in the yield condition, Christiansen et al. presented a strategy for automatic mesh refinement in limit analysis [11]. Lyamin et al. adapted the approach of Borges and developed an adaptive remeshing procedure for lower bound limit analysis [12]. Based on elemental and edge contributions to the bound gap, Munoz et al. construct a new error estimate employed in an adaptive remeshing strategy which is able to reproduce fan-type mesh patterns around points with discontinuous surface loading [13].

In displacement finite-element analysis, triangular elements and quadrilateral elements are fundamentally different. We are aware of that general quadrilateral elements are very frequently encountered in two-dimensional analyses and linear triangular element is less accurate compared to linear quadrilateral element, but in finite-element limit analysis, three-node triangular element is employed exclusively for two-dimensional problems, since this leads to an optimization problem with linear constraints.

In this paper, we extend the quadrilateral element to lower- and upper-bound limit analysis based on the weak form of the equilibrium equations. It is also shown that quadrilateral element is not only more accurate than triangular element, but it converges faster than triangular element as the mesh is refined.

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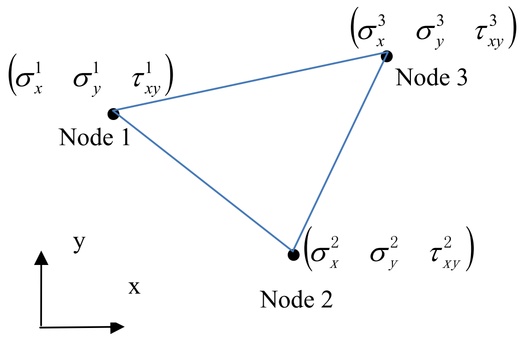

Fig. 1. Three-node linear stress triangle for lower bound limit analysis.

2. Brief review of limit analysis

2.1. Lower bound theorem

Consider a body of volume V with surface area S, if the state stress inside V, σ_{ij}, satisfies equilibrium equations with body forces b_i and the surface tractions T_i acting along the surface S_T, and does not violate the yield criterion $f(\sigma_{ij})$ at any point, then collapse does not occur. That is to say, the applied load is definitely less than or at most equal to the true collapse load. This problem can be reformulated to an optimization problem of the form

$$\text{Max}: \lambda$$
$$\text{S.t.} \begin{cases} \sigma_{ij,j} + \lambda b_i = 0 \quad (\text{in } V) \\ f\left(\sigma_{ij}\right) \leq 0 \quad (\text{in } V) \\ \sigma_{ij} n_j = \lambda \overline{T_i} \quad (\text{on } S_T) . \end{cases} \tag{1}$$

In this paper, the Mohr–Coulomb yield criterion is chosen to model the plastic flow in soils.

2.2. Discrete formulation of the lower bound theorem

Based on a linear three-node triangle, with the unknowns being the stresses at each node, shown in Fig. 1, the variation of the stress throughout each element is linear and each node is associated with 3 unknown stresses σ_X, σ_Y and τ_{XY}, Each stress varies through an element according to

$$\sigma_X = \sum_{i=1}^{3} N_i \sigma_x^i, \qquad \sigma_Y = \sum_{i=1}^{3} N_i \sigma_y^i, \qquad \tau_{XY} = \sum_{i=1}^{3} N_i \tau_{xy}^i \tag{2}$$

where σ_x^i, σ_y^i and τ_{xy}^i are the nodal stresses and N_i, are linear shape functions. These shape functions are

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y) \tag{3}$$

where

$$a_i = x_j y_k - x_k y_j, \qquad b_i = y_j - y_k, \qquad c_i = -x_j + x_k \tag{4}$$

i, j, k are counterclockwise sequence numbers for the three nodes, respectively, and A is the element area.

The lower bound problem can then be stated as a linear programming problem of the form

$$\text{Max}: \lambda$$
$$\text{S.t.} \begin{cases} [A_1]\{\sigma\} = \lambda \{b_1\} \\ [A_2]\{\sigma\} \leq \{b_2\} \end{cases} \tag{5}$$

in which $\{\sigma\}$ is a global vector of unknown nodal stresses, $[A_1]$ is a matrix of all equality constraints and $[A_2]$ is a matrix of yield constraints. The equality constraints include continuum and discontinuity equilibrium and stress boundary conditions, the inequalities represent the linearized yield conditions.

3. Quadrilateral elements for lower bound limit analysis

3.1. Stress approximation

A four-node two-dimensional quadrilateral is shown in Fig. 2.