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Analysis of Galerkin and streamline-diffusion FEMs on piecewise equidistant meshes for turning point problems exhibiting an interior layer

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ABSTRACT

We consider singularly perturbed boundary value problems with a simple interior turning point whose solutions exhibit an interior layer. These problems are discretised using higher order finite elements on layer-adapted piecewise equidistant meshes proposed by Sun and Stynes. We also study the streamline-diffusion finite element method (SDFEM) for such problems. For these methods error estimates uniform with respect to ε are proven in the energy norm and in the stronger SDFEM-norm, respectively. Numerical experiments confirm the theoretical findings.

1. Introduction

We consider singularly perturbed boundary value problems of the form

$$\begin{aligned} -\varepsilon u''(x) + a(x)u'(x) + c(x)u(x) = f(x) \qquad \text{in } (-1,1), \\ u(-1) = \nu_{-1}, \quad u(1) = \nu_1, \end{aligned} \tag{1.1a}$$

with a small parameter $0 < \varepsilon \ll 1$ and sufficiently smooth data a, c, f satisfying

$$a(x) = -(x - x_0)b(x), \qquad b(x) > 0, \qquad c(x) \geq 0, \qquad c(x_0) > 0 \tag{1.1b}$$

for a point $x_0 \in (-1, 1)$. The simple zero x_0 of a is an attractive simple turning point of the problem. Thus, the solution of (1.1) exhibits an interior layer of "cusp"-type [13] at x_0.

In the literature (see e.g. [4], [7, p. 95], [13, Lemma 2.3]) bounds for such interior layers are well known. We have

$$\left|u^{(i)}(x)\right| \leq C\left(1 + \left(\varepsilon^{1/2} + |x - x_0|\right)^{\lambda - i}\right) \tag{1.2}$$

where the parameter λ satisfies $0 < \lambda < \bar{\lambda} := c(x_0)/|a'(x_0)|$. Note that the estimate also holds for $\lambda = \bar{\lambda}$, if $\bar{\lambda}$ is not an integer. Otherwise there is an additional logarithmic factor, see references above. In the following we assume $x_0 = 0$ for convenience.

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The quest for uniform error estimates for singularly perturbed problems has concerned researchers for many years. One of the common strategies is the use of layer-adapted meshes to treat the occurring boundary and interior layers. In particular, meshes for layers of exponential type have been examined, see e.g. [6] where various problems, numerical methods, and meshes are presented. Popular examples are, due to their simplicity, the piecewise equidistant Shishkin meshes [11,9] which are fine only in the layer region. Unfortunately, the layers of "cusp"-type (1.2) do not fade away that quickly and, thus, local refinements do not suffice to capture the layer. Therefore, Sun and Stynes [13, Section 5.1] generalise the standard Shishkin approach and propose a mesh consisting of $\mathcal{O}(\ln N)$ equidistant parts to analyse linear finite elements. Moreover, in [7] Liseikin uses graded meshes adapted to (1.2) to prove the ε-uniform first order convergence of an upwind scheme in the discrete maximum norm.

For problems of the form (1.1) the finite element method is analysed in [1] on the graded meshes of Liseikin. Using related techniques we shall extend the results of Sun and Stynes [13] by studying finite elements of order $k \geq 1$ on piecewise uniform meshes with slightly modified parameters, see Section 3. In particular, we prove ε-uniform error estimates of the form $\left(N^{-1}\ln N\right)^k$ in a weighted energy norm.

In numerical experiments non-physical oscillations in the error can be observed. In order to damp such behaviour various stabilisation techniques have been proposed in recent years. We shall study the streamline-diffusion finite element method (SDFEM) first introduced by Hughes and Brooks [5]. In Section 4 we prove an error estimate in the SDFEM-norm. Moreover, for linear elements a supercloseness result is given which allows to improve the bound for the L^2-norm error. As an example for the analysis in the context of Shishkin meshes we may refer to Stynes and Tobiska [12] who studied a two-dimensional convection-diffusion problem with exponential boundary layers for Q_p-elements.

Some numerical results are given to illustrate the theoretical findings in Section 5.

Notation: In this paper let C denote a generic constant independent of ε and the number of mesh points. Furthermore, for an interval I we use the usual Sobolev spaces $H^1(I)$, $H_0^1(I)$, $W^{k,\infty}(I)$, and $L^2(I)$. The space of continuous functions on I is written as $C(I)$. We denote by $(\cdot,\cdot)_I$ the usual $L^2(I)$ inner product and by $\|\cdot\|_I$ the $L^2(I)$-norm. Moreover, the supremum norm on I is written as $\|\cdot\|_{\infty,I}$ and the seminorm in $H^1(I)$ as $|\cdot|_{1,I}$. If $I=(-1,1)$, the index I in inner products, norms, and seminorms will be omitted. Further notation will be introduced later at the beginning of the sections where it is needed.

2. FEM-analysis on arbitrary meshes

The following section is based on the paper of Sun and Stynes [13]. While their approach merely allows the analysis of linear finite elements, the subsequent results enable the analysis of finite elements of higher order. We will only consider homogeneous Dirichlet boundary conditions $\nu_{-1}=\nu_1=0$. This is no restriction since it can be easily ensured by modifying the right-hand side f. Without loss of generality (cf. [13, Lemma 2.1]) we may assume that

$$\left(c-\tfrac{1}{2}a'\right)(x) \geq \gamma > 0, \qquad \text{for all } x\in[-1,1], \quad \varepsilon \text{ sufficiently small.} \tag{2.1}$$

For $v, w \in H_0^1((-1,1))$ we set

$$B_\varepsilon(v,w) := \left(\varepsilon v', w'\right) + \left(a v', w\right) + (cv, w).$$

Thanks to (2.1) the bilinear form $B_\varepsilon(\cdot,\cdot)$ is uniformly coercive over $H_0^1((-1,1)) \times H_0^1((-1,1))$ in terms of the weighted energy norm $|||\cdot|||_\varepsilon$ defined by

$$|||v|||_\varepsilon := \left(\varepsilon\,|v|_1^2 + \|v\|^2\right)^{1/2}.$$

The weak formulation of (1.1) with $\nu_{-1}=\nu_1=0$ reads as follows:

Find $u \in H_0^1((-1,1))$ such that

$$B_\varepsilon(u,v) = (f,v), \qquad \text{for all } v \in H_0^1((-1,1)). \tag{2.2}$$

Let $-1 = x_{-N} < \ldots < x_i < \ldots < x_N = 1$ define an arbitrary mesh on the interval $[-1,1]$. The mesh interval lengths are given by $h_i := x_i - x_{i-1}$. For $k \geq 1$ we denote by $P_k((x_a,x_b))$ the space of polynomial functions of maximal order k over (x_a,x_b). Furthermore, we define the trial and test space V^N by

$$V^N := \left\{v \in C([-1,1]) : v|_{(x_{i-1},x_i)} \in P_k((x_{i-1},x_i))\,\forall i,\ v(-1)=v(1)=0\right\}.$$

The discrete problem is given by:

Find $u_N \in V^N$ such that

$$B_\varepsilon(u_N, v_N) = (f, v_N), \qquad \text{for all } v_N \in V^N. \tag{2.3}$$

Let u_I denote the standard Lagrangian interpolation into V^N, using the mesh points and $k-1$ (arbitrary) inner interpolation points per interval. For example uniform or Gauß–Lobatto points could be chosen.