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Fractional-order Bernoulli functions and their applications in solving fractional Fredholem–Volterra integro-differential equations

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ABSTRACT

In this paper, we define a new set of functions called fractional-order Bernoulli functions (FBFs) to obtain the numerical solution of linear and nonlinear fractional integro-differential equations. The properties of these functions are employed to construct the operational matrix of the fractional integration. By using this matrix and the least square approximation method the fractional integro-differential equations are reduced to systems of algebraic equations which are solved through the Newton's iterative method. The convergence of the method is extensively discussed and finally, some numerical examples are shown to illustrate the efficiency and accuracy of the method.

1. Introduction

The idea of fractional derivative dates back to a conversation between two mathematicians: Leibniz and L'Hopital. In 1695, they exchanged about the meaning of a derivative of order $\frac{1}{2}$. Their correspondence has been well documented and is stated as the foundation of fractional calculus [26].

The modeling of many real-life physical systems leads to a set of fractional differential equations. Also, fractional order dynamics appear in various physical processes such as, electromagnetic, viscoelasticity [3], waves control theory, bioengineering [20], dynamics of interfaces between nanoparticles and substrates [9], robotics and edge detection and etc. Since most of fractional differential equations do not have exact analytic solution, therefore, many authors have worked on numerical methods for solution of this kind of equations. In recent years, many numerical methods have emerged, such as eigenvector expansion [42], homotopy perturbation method [1], variational iteration method [25], differential transform method [12], Adomian decomposition method [11], finite difference method [41], Laplace transforms method [27], Tau method [36,38], collocation method [35,34], wavelet method [30], fractional-order wavelets [28,31] and so on.

There are three classes of orthogonal functions which are widely used. The first class includes sets of piecewise constant functions (e.g., block-pulse, Haar and Walsh). The second class consists of sets of orthogonal polynomials (e.g., Chebyshev, Legendre and Laguerre) [37]. The third class is the set of sine-cosine functions in the Fourier series. Orthogonal functions have been used in dealing with different problems of the dynamical systems. The main advantage of using orthogonal

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functions is that they reduce the dynamical system problems to those of solving a system of algebraic equations by using the operational matrices of integration or differentiation, thus greatly simplifying the problems. These matrices can be uniquely determined based on the particular orthogonal functions. Special attention has been given to applications of the Walsh function [8], block pulse function [21], Laguerre series [15], Haar function [6], piecewise constant orthogonal functions [39], shifted Legendre polynomials [5], shifted Chebyshev polynomials [40], Legendre wavelet [32] and semi-orthogonal wavelets. The Bernoulli polynomials [17] and Taylor series [22] are not based on orthogonal functions. Nevertheless, they possess the operational matrix of integration. However, since the integration of the cross product of two Taylor series is given in terms of a Hilbert matrix [33], which are known to be ill conditioned, the applications of Taylor series are limited. For approximating an arbitrary time function the advantages of Bernoulli polynomials, over shifted Legendre polynomials on the interval $[0, 1]$, are given in [23].

Recently, in [16], Kazem et al. defined the new orthogonal functions based on the Legendre polynomials to obtain an efficient spectral technique for solving fractional differential equations (FDEs). The paper [44] extended this definition and presented the operational matrix of fractional derivative and integration for such functions to construct a new Tau technique for solving fractional partial differential equations (FPDEs). The authors [4] proposed the fractional-order generalized Laguerre functions based on the generalized Laguerre polynomials. They used these functions to find numerical solution of systems of fractional differential equations. Yuzbasi [45], presented a collocation method based on the Bernstein polynomials for the fractional Riccati type differential equations, by replacing $t \to t^{\alpha}$ in the truncated Bernstein series. Moreover, the authors of [7] expanded fractional Legendre functions to interval $[0, h]$ and to acquire numerical solution of FPDEs.

So, the objective of this paper is to define the new fractional-order functions based on the Bernoulli polynomials for solving the fractional Fredholem–Volterra integro-differential equations. This method is accurate and easy to implement in solving FDEs.

In this work, firstly fractional derivative of the unknown function $D^{\nu}y(x)$ and $y(x)$ in the underlying fractional integro-differential equation are approximated by finite linear combinations of the fractional-order Bernoulli functions (FBFs). Then, we obtain the FBFs operational matrix of fractional integration. Finally, the problem is converted to a system of algebraic equations by using the FBFs operational matrix together with the least square approximation method. Therefore, there are some questions to be answered:

(i) How to derive FBFs operational matrices of the fractional integration and derivative.

(ii) How to analyze the fractional Fredholem–Volterra integro-differential equations via FBFs operational matrix of the fractional integration with the least square approximation method.

(iii) How to select value of fractional order (α) of new functions for different problems.

The outline of this article is as follows. In section 2, we introduce some basic definitions and mathematical preliminaries of the fractional calculus theory. In section 3, the fractional-order Bernoulli functions and their properties are obtained. Section 4, is devoted to construct the FBFs operational matrices of fractional integration and derivative. In section 5, the numerical scheme for solving the fractional Fredholem–Volterra integro-differential equations is expressed. In section 6, we discuss on the convergence of the method presented in section 5. Finally, in section 7, we report our numerical results and demonstrate the accuracy of the proposed method by considering five numerical examples. Section 8 consists of a brief summary.

2. Preliminaries and notations

In this section, we give some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be used in this paper. There are different definitions of fractional integrations and derivatives. The widely used definition of a fractional integration is the Riemann–Liouville definition and of a fractional derivative is the Caputo definition.

Definition 1. The Riemann–Liouville fractional integral operator of order ν is defined as [29]

$$I^{\nu}y(t) = \begin{cases} \frac{1}{\Gamma(\nu)}\int_0^t \frac{y(s)}{(t-s)^{1-\nu}}ds, & \nu > 0,\ t > 0, \\ y(t), & \nu = 0. \end{cases} \tag{1}$$

For the Riemann–Liouville fractional integral we have [45,29]

1. $I^{\nu}(\lambda_1 y(t) + \lambda_2 u(t)) = \lambda_1 I^{\nu}y(t) + \lambda_2 I^{\nu}u(t)$,
2. $I^{\nu_1}I^{\nu_2}y(t) = I^{\nu_1+\nu_2}y(t)$,
3. $I^{\nu_1}I^{\nu_2}y(t) = I^{\nu_2}I^{\nu_1}y(t)$,
4. $I^{\nu}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)}t^{\nu+\beta}, \qquad \beta > -1$,

where λ_1 and λ_2 are constants.