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A modified iterative regularization method for ill-posed problems

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ABSTRACT

In this paper, we study a modified Landweber iteration method via a gradient flow equation induced by a weighted least squares functional. We investigate the proposed scheme for solving ill-posed problems under the setting of compact operator and pseudo-differential operator. The a-priori and the a-posteriori choice rules for regularization parameters are given and both rules yield the order optimal error estimates. Relative to the classical Landweber method, the new method significantly reduces the number of iterations needed to match an appropriate stopping criterion. As applications, we focus on some important ill-posed problems arising from mathematical physics. Numerical experiments are conducted for showing the validity of the scheme.

1. Introduction

In recent years, iterative methods have become more and more sophisticated for solving ill-posed problems, let's refer to [16,31]. For large-scale problems, iterative regularization methods are an attractive alternative to Tikhonov method. In 1951, Landweber [22] proposed a fixed point iterative method for solving the following linear equation which can model many problems arising various applications:

$$Ax = y, \tag{1.1}$$

where A is a bounded linear operator from a Hilbert space X to Hilbert space Y with non-closed range, y is the exact data and y^δ is the observed data with

$$\|y^\delta - y\| \le \delta, \tag{1.2}$$

δ is assumed to be the known error bound, and $x^\dagger$ is the solution which we seek. Landweber iterative scheme for solving (1.1) is given by

$$x_{m+1} = x_m + \beta A^*(y - Ax_m), \ m = 0, 1, \cdots, \tag{1.3}$$

where A^* is the adjoint operator of A, β is the relaxation factor, x_0 is an initial guess and x_m is the m-th step iterative approximate solution. In the context of engineering science, the Landweber iterative method has been found to be equivalent to other methods, for examples, ART and Cimino's method in computerized tomography [17], the Gerchberg–Papoulis

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algorithm for extrapolation of band-limited functions in signal processing [10,24], and the Van Cittert iteration in image reconstruction [21]. It is well-known the Landweber method is an important method in the field of inverse problems and it attracts many researchers. On the accelerated strategy, the readers can refer to [4,7], on the theory and applications, let us refer to the literature [4,15].

However, the classical Landweber method has a defect. That is the approximate solution over-smooths and can not reconstruct the feature of the exact solution, e.g., the jump of the solution. Most of the classical regularization methods such as Tikhonov method suffered from the over-smoothing approximate solutions. In general, in order to reconstruct the feature of the exact solution, two popular ways for dealing with (1.1): one is the least square method with total variation regularization terms or the sparse reconstruction method; the other is the newly-developed fractional regularization methods. The former usually leads to nonlinear algorithms, but the latter leads to linear algorithms and can overcome the disadvantage of over-smoothing to some extent. The fractional regularization methods was first proposed by Klann, Maass and Ramlau in studying a two-step regularization method [20]. Recently the fractional Tikhonov method has been well studied when A is a compact operator with a known singular system. In [19], Klann considered a general fractional regularization method based on filter functions, in [13], Hochstenbach and Reichel presented a modified fractional Tikhonov method for linear discrete ill-posed problem, in [11], Gerth gave some comments and observations on the fractional Tikhonov methods. To overcome the saturation effect (see Section 4.2 in [7]), in [1], Bianchi investigated the iterated fractional Tikhonov methods. The fractional Tikhonov methods have been found to outperform the standard one especially when the problem is severely ill-posed [11]. Until now the theoretical results of most of these works are limited to the following cases: (I) the a-priori regularization parameter choice rules; (II) the operator A is compact. Motivated by the advantage of fractional Tikhonov regularization methods, in this paper, we investigate a new fractional Landweber iteration method which is derived from a gradient flow equation. We will show that the new modified method is of optimal order under the usual a-priori and a-posteriori parameter choice rules when A is a compact operator or a pseudo-differential operator. It is worthy noting that a fractional Landweber iteration method has been provided in [19] where the schemes do not have the same final formulations. The method presented by us is based on the method of gradient flow, please refer to Section 2.

Our contributions are twofold. On the one hand, we make a systematic study on the modified Landweber iteration method including the stationary and non-stationary schemes, the a-priori and a-posteriori parameter choices. On the other hand, the modified method has the advantage that it requires a fewer number of iterations (see Section 6) and it may reduce the over-smoothing property of the classical Landweber method (see Proposition 2.1).

The rest of this paper is organized as follows. In Section 2, we derive the modified Landweber method and consider the regularization effect for overcoming the over-smoothing property; in Section 3, we prove the error estimates when A is a compact operator; in Section 4, we prove the error estimates when A is a pseudo-differential operator; in Section 5, we give some examples and applications; in the last section, we conduct some numerical tests to show the validity of the proposed regularization methods, finally we give some concluding remarks.

2. A modified iterative scheme

Introduce a weighted seminorm [11,13]:

$$\|z\|_W := \|W^{1/2}z\|_Y, \tag{2.1}$$

where $W = (AA^*)^{(\gamma-1)/2}$ for some index $0 \le \gamma \le 1$, and if $\gamma < 1$, W is well-defined by the aid of the Moore–Penrose pseudo-inverse of AA^*. Consider the following least-squares functional

$$J_\gamma(x) = \frac{1}{2}\|Ax - y\|_W^2 = \frac{1}{2}\|(AA^*)^{(\gamma-1)/4}(Ax - y)\|_Y^2, \ \forall x \in X, \tag{2.2}$$

one can easily compute the gradient of the functional (2.2) $\nabla J_\gamma(x) = (A^*A)^{\frac{\gamma+1}{2}}x - (A^*A)^{\frac{\gamma-1}{2}}A^*y$. The corresponding gradient flow is given by

$$\frac{\partial x}{\partial t} = -\nabla J_\gamma(x) = -((A^*A)^{\frac{\gamma+1}{2}}x - (A^*A)^{\frac{\gamma-1}{2}}A^*y). \tag{2.3}$$

With a final time T and an initial guess x_0, this leads to the fractional asymptotic regularization for solving ill-posed problem (1.1). When $\gamma = 1$, it yields the classical asymptotical regularization [7,30]. The gradient flows for corresponding different functionals yield different regularization methods, the readers can consult the reference [2]. In order to solve the abstract evolution equation (2.3) with an initial condition $x(0) = x_0$, all kinds of numerical methods can be applied, e.g., we can use the explicit Euler method with fixed time step length β and get the modified Landweber iteration method:

$$x_m = x_{m-1} - \beta((A^*A)^{\frac{\gamma+1}{2}}x_{m-1} - (A^*A)^{\frac{\gamma-1}{2}}A^*y). \tag{2.4}$$

Therefore, by induction we have