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An error estimate for an energy conserving spectral scheme approximating the dynamic elastica with free ends

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ABSTRACT

An energy conserving spectral scheme is presented for approximating the smooth solution of the dynamic elastica with free ends. The spatial discretization of the elastica is done on the basis of Galerkin spectral methods with a Legendre grid. It is established that the scheme has the unique solution and enjoys a spectral accuracy with respect to the size of the spatial grid. Moreover, some results of a numerical simulation are given to verify that the implemented scheme preserves the discrete energy.

1. Introduction

This paper is concerned with the numerical solution for the system of partial differential equations

$$(\cos\phi)_{tt} = \lambda_{xx}, \tag{1.1}$$

$$(\sin\phi)_{tt} = \nu_{xx}, \tag{1.2}$$

$$\phi_{tt} - \phi_{xx} = \beta(-\lambda\sin\phi + \nu\cos\phi), \tag{1.3}$$

$$0 < x < 1, \quad t > 0,$$

with the unknowns $\phi(x,t)$, $\lambda(x,t)$ and $\nu(x,t)$, and a physical parameter

$$\beta > 0.$$

The system is a normalized form of the dynamic Euler elastica that describes the planer motion of an inextensible elastic beam under geometrically large deflection (see e.g., [1,6] for descriptions and theoretical analyses of the system, and [4] for an extended model describing the three-dimensional motion).

Though there can be several types of boundary conditions according to how the beam is supported, we restrict ourselves to those which have one or more free ends:

⟨FF⟩ free at both ends

$$\lambda(x,t) = \nu(x,t) = \phi_x(x,t) = 0 \quad \text{at } x = 0, 1, \tag{1.4a}$$

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⟨CF⟩ cramped at one end and free at the other

$$\lambda_x(0,t)=\nu_x(0,t)=\phi(0,t)=0, \quad \lambda(1,t)=\nu(1,t)=\phi_x(1,t)=0, \tag{1.4b}$$

⟨HF⟩ hinged at one end and free at the other

$$\lambda_x(0,t)=\nu_x(0,t)=\phi_x(0,t)=0, \quad \lambda(1,t)=\nu(1,t)=\phi_x(1,t)=0. \tag{1.4c}$$

Initial conditions are

$$\phi(x,0)=\phi^0(x), \quad \phi_t(x,0)=\psi^0(x) \tag{1.5}$$

for those data ϕ^0, ψ^0 that are sufficiently smooth, satisfy some compatible conditions.

The purpose of this paper is to present an energy-conserving spectral scheme that approximates the smooth solution of the initial-boundary value problem (1.1)–(1.5). Falk and Xu [5] presented an energy-conserving difference scheme for the problem with a periodic boundary conditions rather than (1.4), and proved that the scheme is convergent at the rate $O(\Delta x^2)$ as the grid spacing $\Delta x=\sqrt{2}\Delta t\to 0$. Moreover, some numerical experiments using their scheme are made in [3]. For the same periodic boundary condition, Ito [8] presented an energy-conserving spectral scheme, and compared the discrete solution with an particular exact solution to indicate that the scheme is convergent at a rate $O(N^{-4})$ as the grid size $N\approx\sqrt{3/\Delta t}$ increases.

For the boundary conditions (1.4), some numerical methods were introduced without evaluations for accuracy. Ito [7] presented an energy-conserving, finite difference scheme to examine the effect of boundary controls on the large deflecting cantilever. Santillan et al. [9,10] treated a model in which the rotational inertia term ϕ_{tt} in (1.3) was neglected, and presented a finite difference scheme to analyze vibrations neighboring equilibrium configurations of the cantilever. In the present paper, we propose an energy-conserving scheme that enjoys the spectral and second-order accuracy with respect to the spatial and temporal discretization, respectively; it is convergent at the rate $O(N^{-m})+O(\Delta t^2)$ if the exact solution $\phi(x,t)$ is of C^m-class for some m.

Before going on the numerical scheme, we formulate the boundary value problem (1.1)–(1.4) in a variational form. Integrating (1.1) and (1.2) twice in x, and using (1.4), we have, for ⟨FF⟩,

$$\lambda(x,t)=\int_0^x\int_0^\eta(\cos\phi)_{tt}\,d\xi d\eta - x\int_0^1\int_0^\eta(\cos\phi)_{tt}\,d\xi d\eta,$$

$$\nu(x,t)=\int_0^x\int_0^\eta(\sin\phi)_{tt}\,d\xi d\eta - x\int_0^1\int_0^\eta(\sin\phi)_{tt}\,d\xi d\eta,$$

and, for ⟨CF⟩ and ⟨HF⟩,

$$\lambda(x,t)=-\int_x^1\int_0^\eta(\cos\phi)_{tt}\,d\xi d\eta, \quad \nu(x,t)=-\int_x^1\int_0^\eta(\sin\phi)_{tt}\,d\xi d\eta.$$

Multiplying (1.3) with any smooth function ψ, integrating the result in x from 0 to 1, and using the above representation of λ and ν, we can arrive at the variational form of the problem:

$$\int_0^1\phi_{tt}\psi\,dx+\int_0^1\phi_x\psi_x\,dx=\beta a\big((\cos\phi)_{tt},\psi\sin\phi\big)-\beta a\big((\sin\phi)_{tt},\psi\cos\phi\big), \tag{1.6}$$

where a is a bilinear form defined by

$$a(f,g)=\int_0^1\Big(\int_0^x f(\xi)\,d\xi\Big)\Big(\int_0^x g(\xi)\,d\xi\Big)\,dx-b(f)b(g)$$

for functions f and g on $[0,1]$, being

$$b(f)=\begin{cases}\displaystyle\int_0^1\int_0^x f(\xi)\,d\xi dx & \text{for ⟨FF⟩}\\ 0 & \text{for ⟨CF⟩ and ⟨HF⟩}\end{cases} \tag{1.7}$$

Note that