Applied Numerical Mathematics 120 (2017) 21–34

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A note on Hermite multiwavelets with polynomial and exponential vanishing moments

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ARTICLE INFO

Article history:
Received 20 December 2016
Received in revised form 6 April 2017
Accepted 26 April 2017
Available online 2 May 2017

Keywords:
Multiwavelets
Hermite subdivision schemes
Factorization
Annihilator

ABSTRACT

The aim of the paper is to present Hermite-type multiwavelets, i.e. wavelets acting on vector data representing function values and consecutive derivatives, which satisfy the vanishing moment property with respect to elements in the space spanned by exponentials and polynomials. Such functions satisfy a two-scale relation which is level-dependent as well as the corresponding multiresolution analysis. An important feature of the associated filters is the possibility of factorizing their symbols in terms of the so-called cancellation operator. This is shown, in particular, in the situation where Hermite multiwavelets are obtained by completing interpolatory level-dependent Hermite subdivision operators, reproducing polynomial and exponential data, to biorthogonal systems. A few constructions of families of multiwavelet filters of this kind are proposed.

1. Introduction

It is well-known that *multiwavelets* generalize classical wavelets in the sense that the corresponding multiresolution analysis (for which we will often use the acronym MRA) is generated by translates and dilates of not just one but several functions. These functions can be assembled in a vector, also known as *multi-scaling function*, satisfying a *vector refinement equation*, whose coefficients are matrices rather than scalars (see [17] for an overview on the topic). Such generalization can result in some advantages connected to the possibility of constructing multiwavelet bases, for example, with short support and a high number of *vanishing moments*. In the scalar situation, the latter is a very crucial and desired property in applications since it reflects on the filters, in the sense that they are able to cancel polynomial discrete data up to a certain degree, thus assuring certain compression capabilities of the overall system associated to the discrete wavelet transform. In the vector setting, nevertheless, it cannot be exploited directly in practical implementations, because the vanishing moments associated to the multiwavelet function do not imply corresponding discrete cancellation properties on the filters side. This results in combining the discrete multiwavelet transform with computationally costly pre-processing and post-processing steps [2,13], unless full-rank filters [12,4,5] or balanced multiwavelets [1,18] are used. Also, except in these cases, no easy factorization of the symbol as in the scalar situation can be considered.

In this paper we are especially interested in multiwavelets of *Hermite-type*, representing a special instance of multiwavelets, where the sequences involved in the transform process are connected not to generic vector data, but to vectors consisting of function values and consecutive derivatives up to a certain order (in this respect, no pre-processing step is

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http://dx.doi.org/10.1016/j.apnum.2017.04.009
0168-9274/

necessary). Such wavelet systems can find applications in many situations where Hermite data are available (for example in problems of motion control) and need to be processed (for example compressed or smoothed).

Our idea is to exploit the close connection between subdivision schemes and wavelet analysis in order to study and construct Hermite multiwavelets from *Hermite subdivision schemes*, so that the underlying multi-scaling function corresponds to the limit function of the subdivision process. Hermite schemes are a particular case of vector subdivision, as they act on vectors representing function values and derivatives (see, for example, [8–10,15,14,20]).

In particular, we focus on Hermite multiwavelet filters which provide not only polynomial but also exponential data cancellation. We thus use a notion of vanishing moment which extends the one usually given, which refers just to polynomials. This generalized property assures compression capabilities of the wavelet system also in the case where the given data exhibit transcendental features. Wavelets possessing such property have already been studied for example in [24] in a scalar framework. The vector context offers the advantage of providing a higher number of vanishing moments together with a short support. Hermite-type multiwavelets allow, in addition, to express the cancellation property as the factorization of the wavelet filter in terms of the so-called *annihilator* or *cancellation operator* introduced in [6] in the context of the study of level-dependent Hermite subdivision schemes. In [6,7] some conditions have been proved connected to the preservation of elements in the (polynomial and exponential) space spanned by $\left\{1, x, \ldots, x^p, e^{\pm\lambda_1 x}, \cdots, e^{\pm\lambda_r x}\right\}$ with $p, r \in \mathbb{N}$. In particular, the preservation property allows the factorization of the subdivision operator in terms of a minimal annihilator.

We show how, given a Hermite subdivision operator based on a level-dependent mask $\boldsymbol{A}^{[n]}$, satisfying the $V_{d,\Lambda}$-spectral condition, in the sense specified later, it is always possible to complete it to a biorthogonal system, where the wavelet filter possesses the desired polynomial/exponential cancellation property. In general, when dealing either with matrix filters or the bivariate case, such completion can be involved, but, as we will show, it can be proved and carried out in a very simple and elegant way starting from subdivision schemes of interpolatory type (as, for example, in [5,11]). Various families of such Hermite multiwavelets can be thus generated by interpolatory Hermite schemes, for example the one provided in [7] and described later in the paper. In this special case, the MRA is associated to level-dependent vector refinable functions which provide a generalization of the well-known Hermite (or finite element) multi-scaling functions proposed, for example, by Strang and Strela in [23]. Some other examples, as we will show, can be considered as well, all of them giving rise, in the limit (i.e. when the frequencies $\lambda_1, \ldots, \lambda_r$ tend to zero), to Hermite multiwavelets with polynomial vanishing moments.

The paper is organized as follows. In Section 2 we fix the notation and present some basic facts about level-dependent (nonstationary) multiresolution analyses of $L^2(\mathbb{R})$ and related discrete wavelet transforms. In Section 3 we provide some details and properties of Hermite subdivision schemes preserving exponential and polynomial data. A strategy for generating Hermite multiwavelets from such schemes is proposed in Section 4, and a factorization result is formulated. In particular, we focus on multiwavelet systems associated to subdivision schemes of interpolatory type. Finally, in Section 5 we give some examples of our construction, based on explicitly given families of interpolatory Hermite subdivision possessing preservation properties. Some conclusions are finally drawn.

2. Preliminaries and basic facts

Let $\ell^r(\mathbb{Z})$ and $\ell^{r\times r}(\mathbb{Z})$, respectively, denote the spaces of all vector-valued and matrix-valued sequences defined on $\mathbb{Z}$.

A *level-dependent MRA* of $L^2(\mathbb{R})$ is defined as the nested sequence $V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$ of spaces each spanned by the dilates and translates of a finite set of functions, which differs from level to level, that is, for $d \in \mathbb{N}$,

$$V_n := \operatorname{span}\{\phi_0^{[n]}(2^n \cdot -k), \ldots, \phi_d^{[n]}(2^n \cdot -k) : k \in \mathbb{Z}\}, \quad n \in \mathbb{N}. \tag{1}$$

Nonstationary MRAs, in the scalar case ($d = 0$), have been introduced, for example, in [3,21].

For each $n \in \mathbb{N}$, such functions can be arranged in a column vector $\boldsymbol{\Phi}^{[n]} := [\phi_0^{[n]}, \phi_1^{[n]}, \ldots, \phi_d^{[n]}]^T$. The dependency of two vector functions at different levels is given in terms of the *level-dependent two-scale-relation*

$$(\boldsymbol{\Phi}^{[n]})^T = \sum_{k\in\mathbb{Z}} (\boldsymbol{\Phi}^{[n+1]})^T (2 \cdot -k) \boldsymbol{A}_k^{[n]}, \tag{2}$$

where the matrix-valued sequence $\boldsymbol{A}^{[n]} := (\boldsymbol{A}_k^{[n]} : k \in \mathbb{Z}) \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ is called the *mask* of $\boldsymbol{\Phi}^{[n]}$.

In a *biorthogonal* setting those functions play the role of the *primal scaling function vectors*. From the point of view of filter banks the masks correspond to *low-pass filters*.

Given a second level-dependent MRA $(\widetilde{V}_n : n \in \mathbb{N})$ generated by $\widetilde{\boldsymbol{\Phi}}^{[n]}$ satisfying

$$(\widetilde{\boldsymbol{\Phi}}^{[n]})^T = \sum_{k\in\mathbb{Z}} (\widetilde{\boldsymbol{\Phi}}^{[n+1]})^T (2 \cdot -k) \widetilde{\boldsymbol{A}}_k^{[n]}$$

for some matrix-valued masks $\widetilde{\boldsymbol{A}}^{[n]} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$, then the functions $\widetilde{\boldsymbol{\Phi}}^{[n]}$ are said to be *dual scaling function vectors* if the following duality relations are satisfied

$$\langle \boldsymbol{\Phi}^{[n]}, \widetilde{\boldsymbol{\Phi}}^{[n]}(\cdot + k)\rangle := \int_{\mathbb{R}} \boldsymbol{\Phi}^{[n]}(x) (\widetilde{\boldsymbol{\Phi}}^{[n]})^T (x + k)\, dx = \delta_{k,0} \boldsymbol{I}, \quad k \in \mathbb{Z}. \tag{3}$$