Applied Numerical Mathematics 120 (2017) 115–124

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Truncated transparent boundary conditions for second order hyperbolic systems

Ivan Sofronov [a,b,*]

[a] Schlumberger, Moscow, Pudovkina 13, Russian Federation
[b] MIPT, Moscow region, Dolgoprudny, Institutskii per. 1, Russian Federation

ARTICLE INFO

Article history:
Received 29 September 2016
Received in revised form 4 May 2017
Accepted 5 May 2017
Available online 11 May 2017

Keywords:
Transparent boundary conditions
Local boundary conditions
Hyperbolic systems
Cylindrical anisotropy
Orthotropic elasticity
Biot poroelasticity

ABSTRACT

In [22] we announced equations for yielding differential operators of transparent boundary conditions (TBCs) for a certain class of second order hyperbolic systems. Here we present the full derivation of these equations and consider ways of their solving. The solutions represent local parts of TBCs, and they can be used as approximate nonreflecting boundary conditions. We give examples of computing such conditions called 'truncated TBCs' for 3D elasticity and Biot poroelasticity

1. Introduction

Transparent boundary conditions (TBCs) for the wave equation were proposed in [18,10] and were further developed, for example, in [20,11,1,25]. Also different aspects of deriving and exploring TBCs are considered in [4,6,12,14,17,26] (note that Th. 2.1 in [26] repeats formulas of kernels derived in [20]).

As it is known, on a circle ($d=2$) or a sphere ($d=3$) TBCs can be written as

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}+\frac{d-1}{2}\frac{c}{r}u-Q^{-1}\{B_k*\}Q\,u=0 \tag{1}$$

where Q and Q^{-1} are the Fourier and inverse Fourier transform operators over the trigonometric ($d=2$) or spherical ($d=3$) basis, and $\{B_k*\}$ denotes the time convolution operator of the k-th Fourier coefficient with kernels $B_k(t)$ derived analytically [20]. In order to provide efficient calculations with reasonable computational recourses the kernels $B_k(t)$ are approximated by sums of exponentials,

$$B_k(t)\approx\sum_{l=1}^{L_k}a_{l,k}e^{b_{l,k}t},\quad \operatorname{Re}b_{l,k}\le 0.$$

* Correspondence to: Schlumberger, Moscow, Pudovkina 13, Russian Federation.
E-mail address: isofronov@slb.com.

http://dx.doi.org/10.1016/j.apnum.2017.05.002
0168-9274/

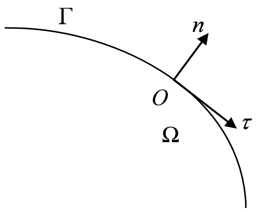

Fig. 1. External boundary Γ, local coordinates at the point O.

Note that the operator of (1) is naturally decomposed into a local (differential) part, the first three terms, which is nothing more than the well-known characteristic boundary conditions for the wave equation, and a nonlocal part in space and time; the latter can be treated as a correction to ensure the absolute accuracy of TBCs.

The explicit analytical expressions for other examples of TBCs operators can be obtained only for simple boundaries (plane, sphere, etc.) and for specific narrow classes of equations, see the discussion in [13]. Extension to the equations with variable coefficients is possible by using the so-called *quasi-analytic* TBCs proposed in [21,23]. Operator coefficients of these conditions are found by solving numerically some auxiliary initial-boundary value problems; therefore, it is a costly computational procedure.

A compromise between the generality of the governing equations that we consider and the accuracy of the boundary conditions is possible if the goal is to derive only the local (differential) part of TBCs and to use it as an operator of local approximate non-reflecting boundary conditions. We call such conditions *truncated* TBCs (TTBCs), as they do not contain the nonlocal part of TBCs. A natural and important generalization of the wave equation is the general *second-order* hyperbolic systems. In [22], we have introduced the equations for evaluating the local part of the TBC operator for such systems. Moreover, we have derived TTBCs for 2D VTI cylindrical anisotropy and for 3D Navier wave equation; a practical application of TTBC for the latter case has been reported in [24].

In the current paper, we provide a detailed derivation of the equations first introduced in the short note [22]. Besides, we present a solution procedure for these equations that yields the operators of the differential part of the TBCs. This is the subject of Section 2.

In Section 3, we consider two additional examples of generating TTBCs: for 3D orthotropic elasticity equations in the cylindrical coordinates, and for 3D Biot poroelasticity equations in the Cartesian coordinates.

Note that for a general hyperbolic system of *first-order* pseudodifferential operators with variable coefficients a way of computing the approximate transparent boundary conditions is considered in [2]. Taking advantage of the recursive formulation of Taylor's method, the authors derive a recursive formula expressing the m^{th} homogeneous symbol of the nonlocal pseudodifferential operator of transparent conditions. In particular, applying to Maxwell's equations they derive a local condition that they call "a second-order complete radiation condition". In [3], the authors consider the time dependent 2D Klein–Gordon equation and derive a set of absorbing boundary conditions using microlocal analysis and pseudodifferential operators. It is an interesting question of whether the techniques developed in [2] and [3] can be applied to the second-order hyperbolic systems we are considering hereafter.

2. Derivation of equations for operators of TTBCs

2.1. Problem formulation

Consider a domain $\Omega \subset R^3$ with a sufficiently smooth open boundary Γ at the point O. Denote by n the outward normal and by $\tau = (\tau_1, \tau_2)$ the vector of orthogonal coordinates in a tangent plane, see Fig. 1.

Let a vector function $u = (u^1, u^2, \dots, u^N)^{\mathrm{T}}$ satisfies a second-order hyperbolic system of equations [9] written in local coordinates (n, τ) at the point O in the form

$$-\frac{\partial^2 u}{\partial t^2} + A\frac{\partial^2 u}{\partial n^2} + (B\nabla_\tau)\frac{\partial u}{\partial n} + C\frac{\partial u}{\partial n} + a(\nabla_\tau, \nabla_\tau)u + (c\nabla_\tau)u + du = 0. \tag{2}$$

Here we use the following notation:

$$(B\nabla_\tau)u \equiv \sum_{i=1}^{2} B_i \frac{\partial u}{\partial \tau_i}, \qquad a(\nabla_\tau, \nabla_\tau)u \equiv \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial \tau_i \partial \tau_j}.$$

The $N \times N$ matrix coefficients $A(\tau)$, $B_i(\tau)$ and $C(n,\tau)$, $a_{ij}(n,\tau)$, $c_i(n,\tau)$, and $d(n,\tau)$ may depend on τ and on n, τ, respectively.

We suppose that the coefficients and solution of (2) are sufficiently smooth in the neighborhood of the point O.

Our goal is to obtain a relationship between the solution u and its first derivatives, which is approximately valid at the point O. This relationship will have the form