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Stability and error analysis of the reproducing kernel Hilbert space method for the solution of weakly singular Volterra integral equation on graded mesh

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A B S T R A C T

In this article, we approximate the solution of the weakly singular Volterra integral equation of the second kind using the reproducing kernel Hilbert space (RKHS) method. This method does not require any background mesh and can easily be implemented. Since the solution of the second kind weakly singular Volterra integral equation has unbounded derivative at the left end point of the interval of the integral equation domain, RKHS method has poor convergence rate on the conventional uniform mesh. Consequently, the graded mesh is proposed. Using error analysis, we show the RKHS method has better convergence rate on the graded mesh than the uniform mesh. Numerical examples are given to confirm the error analysis results. Regularization of the solution is an alternative approach to improve the efficiency of the RKHS method. In this regard, an smooth transformation is used to regularization and obtained numerical results are compared with other methods.

1. Introduction

This work is intended as an attempt to investigate the solution of the weakly singular Volterra integral equation using RKHS method. The second kind Volterra integral equation is presented by

$$u(x)+\int_0^x \frac{K(x,t)}{(x-t)^{\alpha}}u(t)\mathrm{d}t = f(x), \quad x\in[0,1], \tag{1.1}$$

where $K(x,t)$ denotes a continuous function on $D=\{(x,t): 0\le t\le x\le 1\}$, and $f(x)$ is a given continuous function. Specially, when $K(x,x)\neq 0$ and $0<\alpha<1$, (1.1) is said to be weakly singular.

Various phenomena in different fields of science and engineering can be formulated by integral equations [28]. Volterra integral equations appear in numerous applications such as reaction–diffusion study [20], modeling viscoelastic stress in materials [7], modeling of dynamical systems [31], queueing problems [19], etc.

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A singular integral equation is difficult to be solved analytically. Therefore, this entails that its solution to be obtained approximately. In the recent years, RKHS method has gained growing interest in approximate solutions of the different problems. This method has effective and attractive implementation features. Some major advantages of RKHS method are that using different kind of inner product spaces, stability and error analysis can be carried out in various procedures. The obtained approximate solution and its derivatives in the reproducing kernel space converge uniformly to the corresponding exact solution and derivatives.

The theory of reproducing kernel was used for the first time by S. Zaremba on boundary value problems [33]. In 1943, Aronszajn [2] developed the theory of reproducing kernels and introduced the Bergman kernel functions. Recently Cui and Lin [11], after summarizing RKHS theory, have solved some various problems. In [17], Geng et al. solved singularly perturbed turning point problems based on the asymptotic expansion technique and the RKHS method. In [18], Hon and Takeuchi used discretized Tikhonov regularization by RKHS in the solution of backward heat conduction problem. In [3], Beyrami et al. considered the solution of Cauchy singular integral equation by RKHS method using operator transformation. In the conventional RKHS method, image space of the associated operator for the considered problem is a reproducing kernel space, whereas in various situations for singular equations, this condition is substituted with other conditions like subset of continuous or integrable functions [10,15].

Comprehensive problems have been solved using this method such that wide spectrum variety of ordinary or partial differential and integral equations have been solved with this method [16,30]. But error and stability analysis are considered rarely. In Section 4.1 of [11], and also [13,14] stability of RKHS method for the solution of Fredholm integral equation was considered. In [24] stability of linear Fredholm integro-differential equations was investigated. While the first kind of Fredholm integral equation in $L^2[a,b]$ and $C[a,b]$ is ill-posed and do not has unique solution [25], in the reproducing kernel space it becomes a well-posed problem and the obtained solution form RKHS method corresponds to the minimum norm solution. In the mentioned references, it has been stated that RKHS method is stable. In the above references, kernels of integral equations are regular. In [21] and [12], Fredholm integral equations with weakly singular and Cauchy type singular kernel are considered. About error analysis, works conducted by Li and Wu [23] and Ketabchi et al. [22] can be mentioned which considered problems are ordinary differential equation of the second order and Volterra integral equations with regular kernels, respectively. Error analysis of RKHS method for singular Volterra integral equation has not been studied yet.

In this work, we investigate the stability and error analysis of RKHS method for the solution of the weakly singular Volterra integral equation. Some numerical examples are presented in order to validate the error analysis theoretical results about the convergence rate of the approximate solution. In the numerical examples, RKHS method is combined with smooth transformation and obtained results are compared with results of other methods. We also discuss practical implementation of the method and disclose its Mathematica source code.

This paper is organized as follows. In the second section, RKHS is introduced. In Section 3 acquirement of the approximate solution in the RKHS is described. In Sections 4 and 5, error and stability analysis of approximate solution are considered, respectively. Section 6 deals with the numerical examples of weakly singular Volterra integral equations. Section 7 is dedicated to a brief conclusion. Some useful complementary discussion are given in the Appendices.

2. Reproducing kernel space

In this section, in order to solve (1.1), we first describe the reproducing kernel space and recall some basic concepts in the reproducing kernel space.

Definition 2.1. The space $W^m[0,1]$ is defined as follows

$$W^m[0,1] = \{u(x) \mid u^{(m-1)}(x) \in AC[0,1],\ u^{(m)}(x) \in L^2[0,1]\},$$

where $AC[0,1]$ stands for an absolutely continuous real-valued function on $[0,1]$. The inner product and norm in $W^m[0,1]$ are respectively defined by

$$(u,v)_m = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)\mathrm{d}x,$$

and

$$\|u\|_m = \sqrt{(u,u)_m}, \qquad u,v \in W^m[0,1].$$

For distinct values of m, $W^m[0,1]$ is a complete Hilbert space and contains a unique reproducing kernel function [11]. Consequently, $W^m[0,1]$ is called a reproducing kernel Hilbert space. Let $R_x^m(y)$ denote the reproducing kernel of $W^m[0,1]$. For every $u \in W^m[0,1]$ and fixed $x \in [0,1]$, $R_x^m(y)$ satisfies the following reproducing property

$$u(x) = (u(y), R_x^m(y))_m.$$