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Strong convergence of the split-step theta method for neutral stochastic delay differential equations ☆

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ABSTRACT

Neutral stochastic delay differential equations often appear in various fields of science and engineering. The aim of this article is to investigate the strong convergence of the split-step theta (SST) method for the neutral stochastic delay differential equations, where the corresponding coefficients may be highly nonlinear with respect to the delay variables. In particular, we reveal that the SST method with $\theta \in [0, 1]$ strongly converges to the exact solution with the order $\frac{1}{2}$. Some numerical results are presented to confirm the obtained results.

1. Introduction

The theory, computation and applications of stochastic differential equations (SDEs) including stochastic delay differential equations (SDDEs) have been extensively discussed in the recent years (see e.g. [9,16,13,19,28,24]). More recently, some researchers have given special interests to the study of the equations in which the delay argument occurs in the derivative of the state variable, which are called neutral stochastic delay differential equations (NSDDEs). NSDDEs play an important role in modeling some phenomena in science and engineering such as physics, finance, neural network and mathematical biology (see e.g. [3,10,18,20,24,29]).

Like deterministic neutral delay differential equations (NDDEs) and SDEs, most of NSDDEs cannot be analytically solved, so numerical methods have become one of the powerful techniques. There is extensive literature on numerical schemes for NDDEs (see e.g. [2,1,11,17,22] and references therein). Meanwhile, many results have been obtained for strong convergence of the numerical methods for SDEs and SDDEs. Mao [24] and Kloeden & Platen [19] discussed the strong convergence of the Euler–Maruyama (EM) method for the SDEs with the drift and diffusion coefficients satisfying the linear growth and global (or local) Lipschitz conditions, respectively. In Higham, Mao & Stuart [12] and Bastani & Tahmasebi [5], the strong convergence rates of the Euler-type methods were shown for the SDEs with the drift coefficient satisfying one-sided Lipschitz condition and the diffusion coefficient satisfying the global Lipschitz condition. Zhou [36] discussed the strong convergence of the backward Euler–Maruyama (BEM) scheme for the SDEs with the highly nonlinear condition. Ding, Ma & Zhang [7] and Cao, Hao & Zhang [6] considered a class of split-step theta (SST) method and obtained the strong convergence rate in

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mean-square sense for SDEs and SDDEs, respectively. In Huang [14,15], another type of SST method which is different from the SST method in [6,7] was introduced for SDEs and SDDEs, respectively. Bao & Yuan [4] investigated the convergence rate of EM scheme for SDDEs, where the corresponding coefficients may be highly nonlinear with respect to the delay variables. Under the local Lipschitz condition and a coupled condition on the drift and diffusion coefficients, Yue & Huang [33] proved the strong convergence of the SST method in [14] for the SDDEs. In Zhang, Gan & Hu [35], the strong convergence rate of the split-step backward Euler (SSBE) method for the linear SDDEs was obtained, and the SSBE method was improved for SDDEs in [30].

However, only few efforts have been made to solve numerically NSDDEs. In the analysis for strong convergence of numerical methods, a widely used assumption was that the drift and diffusion coefficients satisfy the global Lipschitz and the linear growth conditions. For instance, Zhang & Gan [34] investigated the strong convergence of one-step methods for the NSDDEs with the drift and diffusion coefficients satisfying the global Lipschitz and linear growth conditions, respectively. Gan, Schurz & Zhang [8] proposed the drift-implicit stochastic θ-methods for NSDDEs with the Lipschitz continuous and linear growth coefficients and obtained the convergence order in mean square sense.

Unfortunately, many real-life phenomena are described by NSDDEs with the coefficients which violate the assumption, for example, some NSDDEs appeared in the chemical engineering systems, the theory of aeroelasticity and the neural networks etc. do not satisfy the linear growth condition (see e.g. [10,21,23,25]). Now, we consider a NSDDE on $\mathbb{R}$,

$$d[x(t)-\frac{1}{2}x(t-\tau)]=[ax(t)+bx^3(t-\tau)]dt+[cx(t)+ex^2(t-\tau)]dw(t), \tag{1.1}$$

where $a,b,c,e\in\mathbb{R}$, $\tau>0$ are constants, and $w(t)$ is a scalar Brownian motion. It is easy to observe that both the drift coefficient and the diffusion coefficient are highly nonlinear, especially with respect to the delay arguments. Therefore, the existing convergence results, e.g., [8,31,34], cannot be applied to Eq. (1.1), and the convergence rate of the corresponding schemes also cannot be obtained. In order to weaken this assumption, Zhou & Wu [38] considered the convergence of the EM method for the NSDDEs with Markovian switching under the local Lipschitz condition. In Wu & Mao [31], the strong mean-square convergence theory of the EM method was shown for the neutral stochastic functional differential equations (NSFDEs) under the local Lipschitz and linear growth conditions. In Zhou & Fang [37] and Milošević [25–27], the convergence in probability of the EM, SSBE and BEM methods for NSDDEs under the highly nonlinear conditions were studied, that is, without the linear growth condition, respectively. Nevertheless, the new problem is that the strong convergence rate have not been still revealed in [25–27,37]. Motivated by this consideration, in this paper, we will investigate further the numerical solutions of NSDDEs.

In this paper, we will apply the SST method in Huang [14] to NSDDE whose drift coefficient and diffusion coefficient are highly nonlinear with respect to the delay arguments,

$$d[x(t)-N(x(t-\tau))]=f(x(t),x(t-\tau))dt+g(x(t),x(t-\tau))dw(t), \quad t\in[0,T] \tag{1.2}$$

with initial data $x(t)=\varphi(t)$, $t\in[-\tau,0]$. Here, $\varphi(t)$ is assumed to be continuous and $(\mathfrak{F}_0,\mathbb{B}^n)$-measurable with $\mathbb{E}\big[\sup_{-\tau\le t\le 0}|\varphi(t)|^p\big]<\infty$, $p\ge 2$, in other words, $\varphi(t)\in C^b_{\mathfrak{F}_0}([-\tau,0];\mathbb{R}^n)$, where T, $\tau>0$ are constants, and

$$f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n, \quad g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times d}, \quad N:\mathbb{R}^n\to\mathbb{R}^n$$

be all Borel measurable real-valued functions and $w(t)$ is a d-dimensional Brownian motion. In the following, our main aims will be not only to investigate the strong convergence of the SST scheme for the NSDDEs, which may be highly nonlinear with respect to the delay variables, but also to obtain the convergence rate of this numerical method.

The paper is organized as follows. In section 2, some notations and preliminaries are introduced. In section 3, we present moment estimations of the SST scheme. In section 4, our main results which state the strong convergence rate of the SST method are given. In section 5, three numerical experiments illustrate the obtained results. In the last section, we give a short conclusion.

2. Preliminary notations and split-step theta method

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace (or Frobenius) norm in $\mathbb{R}^{n\times d}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A|=\sqrt{trace(A^TA)}$. $a\vee b$ represents $\max\{a,b\}$ and $a\wedge b$ denotes $\min\{a,b\}$. Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathfrak{F}_0$ contains all $\mathbb{P}$-*null* sets. $w(t)=(w_1(t),w_2(t),\cdots,w_d(t))^T$ is supposed to be a standard d-dimensional Brownian motion defined on the probability space $(\Omega,\mathfrak{F},\mathbb{P})$ with mutually independent coordinates w_i. Throughout the paper, $C>0$ denotes a generic constant whose value may vary with each appearance.

To guarantee the existence and uniqueness of solutions and investigate the convergence rates of their approximations, we introduce the following conditions. Let $V_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ such that

$$0\le V_i(x,y)\le K_i(1+|x|^{q_i}+|y|^{q_i}), \quad i=1,2 \tag{2.1}$$

for some $K_i>0, q_i\ge 1$ and arbitrary $x,y\in\mathbb{R}^n$. We further assume that $f(x,y)$, $g(x,y)$ and $N(x)$ satisfy the following assumptions.