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# Parallel finite element variational multiscale algorithms for incompressible flow at high Reynolds numbers ☆

Yueqiang Shang [a,*], Jin Qin [b]

[a] School of Mathematics and Statistics, Southwest University, Chongqing 400715, PR China
[b] School of Mathematics, Zunyi Normal college, Zunyi 563002, PR China

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ABSTRACT

Based on two-grid discretizations, some parallel finite element variational multiscale algorithms for the steady incompressible Navier–Stokes equations at high Reynolds numbers are presented and compared. In these algorithms, a stabilized Navier–Stokes system is first solved on a coarse grid, and then corrections are calculated independently on overlapped fine grid subdomains by solving a local stabilized linear problem. The stabilization terms for the coarse and fine grid problems are based on two local Gauss integrations. Error bounds for the approximate solution are estimated. Algorithmic parameter scalings are also derived. The theoretical results show that, with suitable scalings of the algorithmic parameters, these algorithms can yield an optimal rate of convergence. Numerical results are given to verify the theoretical predictions and demonstrate the effectiveness of the proposed algorithms.



## 1. Introduction

In this paper, we consider the following incompressible Navier–Stokes equations:

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

which describe the steady flow of an incompressible viscous Newtonian fluid, where $\Omega$ is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ in $\mathbb{R}^d$ ($d = 2$ or 3), $u : \Omega \to \mathbb{R}^d$ is the velocity, $p : \Omega \to \mathbb{R}$ the pressure, $f : \Omega \to \mathbb{R}^d$ the prescribed body force, and $\nu$ the kinematic viscosity. Given a characteristic length $L$ and a characteristic velocity $U$, the Reynolds number is defined as $Re = UL/\nu$.

Numerical solution of the above Navier–Stokes system is at the heart of many engineering applications, such as the vehicle design, the nuclear reactor safety evaluation, and the weather prediction. Among successful numerical methods for the Navier–Stokes equations, two-level or multilevel finite element methods pioneered by Xu [68,69] attracted considerable

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* Corresponding author.
*E-mail address:* yqshang@swu.edu.cn (Y.Q. Shang).

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attention from researchers. The basic idea of two-level finite element methods is to first solve a nonlinear Navier–Stokes problem on a coarse grid, and then solve a linear problem on a fine grid to correct the approximate solution. The two-level finite element methods are very efficient in computations compared with the standard one-level finite element methods. They can yield an approximate solution with an accuracy comparable to that of the solution obtained from the standard one-level finite element methods using the same fine grid. However, due to the fact that only a linear problem needs to be solved on the fine grid, two-level finite element methods can save a large amount of CPU time compared with the standard one-level finite element methods. We refer to, for example, Layton et al. [36,37,39–41], Girault and Lions [7], Liu and Hou [45], and He et al. [17,21,24,43] for two-level or multilevel finite element methods for the steady Navier–Stokes equations, and Girault and Lions [8], Olshanskii [50], He et al. [13,14,18,20], Hou and Mei [29], and Abboud et al. [1] for two-level or multilevel finite element methods for the unsteady Navier–Stokes equations.

Combining the ideas of two-level finite element methods with the local and parallel finite element computation approach of Xu and Zhou [70,71], some parallel two-level finite element algorithms for the Navier–Stokes equations were also proposed and studied by He et al. [19,22], Ma et al. [47,48] and Shang et al. [54,58,60,61], among others. Numerical tests showed that this type of two-level finite element methods are very attractive in simulation of laminar flows. However, due to the fact that a nonlinear Navier–Stokes system needs to be solved on a coarse grid, applications of two-level finite element methods to the simulation of high Reynolds number flows are still challenging (cf. [46,56,62]). Stabilizations are essential in such applications.

In this paper, we present and study some parallel two-level stabilized finite element algorithms for the Navier–Stokes equations at high Reynolds numbers. These algorithms use a variational multiscale model to stabilize the Navier–Stokes system on both coarse and fine grids. Unlike the usual projection-based stabilization techniques (cf. [3,6,32,34,35]), the finite element variational multiscale model employed here is based on two local Gauss integrations. The idea of two local Gauss integrations can be traced back to Li and He [15,42] who used a stabilization term based on two local Gauss integrations to stabilize $P_1$–$P_1$ finite element approximation of the Stokes and Navier–Stokes equations. Subsequently, Zheng et al. [74] used this two local Gauss integrations idea to develop a finite element variational multiscale method for the steady Navier–Stokes equations. Compared with the common projection-based stabilization methods (cf. [3,6,32,34,35]), this stabilization method is free of extra storage, and only needs single-level mesh in computations. We refer to [74] for detailed information about this method. This finite element variational multiscale method was then applied to the time-dependent Navier–Stokes equations [57], the stationary conduction–convection problem [31], and combinized with the adaptive technique [64,73], the two-grid methods [44,63], the domain decomposition methods [55], and the partition of unity [67].

Specifically, in the parallel two-level stabilized finite element algorithms developed in this paper, we first solve a global stabilized nonlinear Navier–Stokes problem on a coarse grid, and then solve a local stabilized linear residual problem on overlapped fine grid subdomains to correct the coarse grid solution in a parallel manner, where the stabilization terms for the coarse and fine grid problems, with different stabilization parameters, are defined by the difference between a consistent and an under-integrated matrix of the gradient of velocity interpolants. The significant feature of the proposed algorithms is that they maintain the best algorithmic features of two-level finite element methods and the variational multiscale method based on two local Gauss integrations. On the one hand, they are able to simulate high Reynolds number flows without spurious oscillations compared with the standard two-level finite element methods. On the other hand, they can save a large amount of computational time compared with the one-level variational multiscale methods.

An outline of the paper is as follows. In the next section, some mathematical preliminaries are given. In Section 3, three local finite element variational multiscale algorithms based on two-grid discretizations are presented and analyzed. Section 4 is devoted to the parallel finite element variational algorithms together with the corresponding theoretical analysis. Some further remarks and discusses about the proposed algorithms are given in Section 5. Numerical results which verify the theoretical predictions and demonstrate the effectiveness of the proposed algorithms are presented in Section 6, followed by concluding remarks in Section 7.

## 2. Mathematical preliminaries

For a nonnegative integer $k$, we denote by $\|\cdot\|_{k,\Omega}$ the usual norm of Sobolev space $H^k(\Omega)^d$ and by $(\cdot,\cdot)$ the standard inner-product of $L^2(\Omega)$ or $L^2(\Omega)^d$ (cf. [2]), and $H_0^1(\Omega)^d=\{v\in H^1(\Omega)^d : v|_{\partial\Omega}=0\}$, where $v|_{\partial\Omega}=0$ is in the sense of trace. The space $H^{-1}(\Omega)^d$, the dual of $H_0^1(\Omega)^d$, and the corresponding norm $\|\cdot\|_{-1,\Omega}$ will also be used. For a subdomain $\Omega_0\subset\Omega$, we view $H_0^1(\Omega_0)^d$ as a subspace of $H_0^1(\Omega)^d$ by extending the functions in $H_0^1(\Omega_0)^d$ to be functions in $H_0^1(\Omega)^d$ with zero outside of $\Omega_0$. For $D\subset\Omega_0\subset\Omega$, we use the notation $D\subset\subset\Omega_0$ to mean that $\mathrm{dist}(\partial D\backslash\partial\Omega,\partial\Omega_0\backslash\partial\Omega)>0$. Throughout this paper, we shall use the letter $c$ or $C$ (with or without subscripts) to denote a generic positive constant which is independent of the mesh parameter and may take on different values on different occurrences.

### 2.1. Functional setting of the Navier–Stokes equations

To give the variational form of the Navier–Stokes equations, we introduce the following Hilbert spaces:

$$X=H_0^1(\Omega)^d,\;\; Y=L^2(\Omega)^d,\;\; M=L_0^2(\Omega)=\{q\in L^2(\Omega): \int_\Omega q\,dx=0\},$$