Applied Numerical Mathematics 117 (2017) 50–68

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Fully-geometric mesh one-leg methods for the generalized pantograph equation: Approximating Lyapunov functional and asymptotic contractivity ☆

Wansheng Wang

School of Mathematics and Statistics, Changsha University of Science and Technology, Yuntang campus, 410114, Changsha, China

ARTICLE INFO

Article history:
Received 24 July 2016
Received in revised form 28 November 2016
Accepted 25 January 2017
Available online 9 February 2017

Keywords:
Neutral functional differential equation
Generalized pantograph equation
Fully-geometric mesh one-leg methods
Variable stepsizes one-leg methods
Asymptotic contractivity
Approximating Lyapunov functional
Stability

ABSTRACT

Motivated by recent stability results on one-step methods, especially Runge–Kutta methods, for the generalized pantograph equation (GPE), in this paper we study the stability of one-leg multistep methods for these equations since the one-leg methods have less computational cost than Runge–Kutta methods. To do this, a new stability concept, $G_q(\bar{q})$-stability defined for variable stepsizes one-leg methods with the stepsize ratio q which is an extension of G-stability defined for constant stepsizes one-leg methods, is introduced. The Lyapunov functional of linear system is obtained and numerically approximated. It is proved that a $G_q(\bar{q})$-stable fully-geometric mesh one-leg method can preserve the decay property of the Lyapunov functional for any $q \in [1, \bar{q}]$. The asymptotic contractivity, a new stability concept at vanishing initial interval, is introduced for investigating the effect of the initial interval approximation on the stability of numerical solutions. This property and the bounded stability of $G_q(\bar{q})$-stable one-leg methods for linear and nonlinear problems are analyzed. A numerical example which further illustrates our theoretical results is provided.

1. Introduction

We are interested in stability properties of fully-geometric mesh (FGM) one-leg methods (OLMs) for the numerical solution of the initial-value problems for neutral functional differential equation with proportional delay, the so-called generalized pantograph equation (GPE), which is an equation of the form

$$\begin{cases} y'(t) = f(t, y(t), y(\lambda t), y'(\lambda t)), & t \in J_T := [0, T], \\ y(0) = \phi, \end{cases} \tag{1.1}$$

where $T > 0$ is a real constant, $\lambda \in (0, 1)$, $\phi \in \mathbf{C}^d$ and $f : [0, +\infty) \times \mathbf{C}^d \times \mathbf{C}^d \times \mathbf{C}^d \to \times \mathbf{C}^d$ is a given mapping. Note that this equation includes as an important special case the non-neutral pantograph equation

$$\begin{cases} y'(t) = f(t, y(t), y(\lambda t)), & t \in J_T, \\ y(0) = \phi. \end{cases} \tag{1.2}$$

☆ This work was supported by the Natural Science Foundation of China [Grant No. 11371074]; the Hunan Provincial Natural Science Foundation of China [Grant No. 13JJ1020]; and the Research Foundation of Education Bureau of Hunan Province, China [Grant No. 13A108].

E-mail address: w.s.wang@163.com.

http://dx.doi.org/10.1016/j.apnum.2017.01.019
0168-9274/

The equations of the form (1.1) are found in a wide range of applications; see, e.g., [26,30] and [13], and the references therein. Since their analysis is often difficult, it is very important to study their numerical approximations in order to explore their solutions. In recent years, the numerical solution of (1.1) has been extensively investigated and many interesting results have been reported (see, e.g., [8–10,31,27,2,29,18,22,45,48,23,49,38,34,47,39,24,40,42,3,33]). For the recent monograph relevant to the numerical solution of GPE, we refer the reader to [5] and [6]. These results are mainly involved in three different discretization approaches, the approach to directly discretizing it on a uniform mesh (UMD, see, e.g., [8–10, 27]), the approach to transforming it into neutral equation with constant delay (TRA, see, e.g., [31,29,22,45,48,39,24,40]), which was suggested by Jackiewicz [28], and the approach to directly discretizing it on a geometric mesh (GMD, see, e.g., [2,18,23,49,38,34,39,41]).

Our aim in this paper is to look for FGM OLMs with stability properties. More specifically, we seek OLMs which produce bounded solution when applied to linear and nonlinear GPE using a geometric step size sequence. It has come to our attention that many researchers mainly considered the one-step methods, especially Runge–Kutta methods, for solving GPE. Since the multistep methods have generally less computational cost than Runge–Kutta methods, some authors (see Zhang and Sun [45], Huang and Gan [22] and Huang [24]) considered transforming (1.1) into neutral delay differential equation with constant delay and discretizing the equation by multistep methods. But, to the best of our knowledge, no work has been carried out on the direct discretization of GPE by multistep methods on a geometric mesh. Although in an abstract sense the approach to TRA and the approach to GMD may be considered to be essentially same [29], there is real difference between them in numerically computing [39]. It is well known that the second order backward differentiation formula (BDF) with constant step size is stable for stiff ODEs. However, its variable step sizes counterpart may be unstable even for nonstiff variable-coefficient ODEs (see, e.g. [36,16,15,20]). Consequently, although Huang and Gan [22], and Huang [24] obtained the asymptotic stability results of general linear methods, including OLMs, connecting with the approach to TRA for GPE, it is not clear whether the variable step sizes counterparts of these methods are stable for GPE on an FGM. Hence, it is interesting and necessary to study the stability of OLMs for neutral equation on an FGM.

We also noticed that the papers [22,24] considered only the asymptotic properties of numerical solution to GPE (1.1) on infinite intervals. Our major concern in this paper is the bounded stability and little attention is paid on the asymptotic stability which implies the stepsize sequence will approach to infinity when an FGM is used, since from a viewpoint of practical computation it is desirable to give more precise information on the behavior of each solution including that when the stepsizes are not so large. Not only that, we study the asymptotic behavior of the numerical solutions at vanishing initial interval and the minimal stepsize. This useful asymptotic property will be called asymptotic contractivity. We will provide some asymptotic results on FGM OLMs for linear and nonlinear GPEs, including ODEs and non-neutral pantograph equations as special cases.

Historically, the variable mesh, including fully-geometric mesh, OLMs, especially variable mesh BDFs, have been extensively applied to various problems (see, e.g., [1,12]), and the stability of variable mesh OLMs for these problems has been thoroughly investigated (see, e.g., [37,15,17,20]). In these classical papers, some important results have been reported, for example, some OLMs are stable for certain stable variable-coefficient problems with constant stepsize but unstable with variable-steps (see, e.g., [37,15]). In this paper we will investigate the stability properties of FGM OLMs for GPE.

To accomplish this we have to review the FGM OLMs and establish preliminary results in Section 2. In this section, a new stability concept, $G_q(\bar{q})$-stability, is introduced and will play important role in the following numerical stability analysis. We also introduce another stability concept, asymptotic contractivity, in this section. In Section 3 we consider the stability of FGM OLMs for solving linear systems, and obtain a Lyapunov functional and its numerical approximation. We extend the study to nonlinear problems and obtain a few results on both asymptotic contractivity and bounded stability in Section 4. Numerical experiments which confirm our theoretical results are given in Section 5. Finally, we give some concluding remarks in Section 6.

2. FGM OLMs: Fully-geometric mesh one-leg methods

In this section, we will introduce the fully-geometric mesh one-leg methods and apply them to generalized pantograph equation (1.1).

2.1. Geometric mesh

For given $\mathrm{N} \in \mathbf{N}$, let $J^{\mathrm{N}} : \{0 = t_0 < t_1 < \cdots < t_n < \cdots < t_{\mathrm{N}} = T\}$ denote a mesh for the given interval J_T. A mesh J^{N} in J_T is called geometric with N steps (t_{n-1}, t_n), $n = 1, 2, \cdots, \mathrm{N}$, and grading factor $r \in (0, 1)$, if

$$t_0 = 0, \qquad t_n = r^{\mathrm{N}-n} T, \quad 1 \le n \le \mathrm{N}.$$

For such a geometric mesh, it is easy to derive that $h_{n+1} = r^{-1} h_n = q h_n, n \ge 2$, when the stepsizes h_n are defined by $h_n := t_n - t_{n-1}$ $(n \ge 1)$. This kind of mesh has been widely used in numerically solving some practice problems such as the differential equations with proportional delay (see, e.g., [2,18,23,49,38,34,39,41]), the integral equations with proportional delay (see, e.g., [21,7]), and problem where singularities may arise at $t = 0$ (see, e.g., [43,35]).