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Analysis of order reduction when integrating linear initial boundary value problems with Lawson methods ☆

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In this paper, a thorough analysis is given for the order which is observed when integrating evolutionary linear partial differential equations with Lawson methods. The analysis is performed under the general framework of C_0-semigroups in Banach spaces and hence it can be applied to the numerical time integration of many initial boundary value problems which are described by linear partial differential equations. Conditions of regularity and annihilation at the boundary of these problems are then stated to justify the precise order which is observed, including fractional order of convergence.

1. Introduction

Exponential methods have very much been developed and analyzed in the literature in order to integrate partial differential equations [12,16]. As the system which arises after space discretization is stiff, exponential methods are a valuable tool because they are able to integrate it in an explicit and stable way.

In this paper, we will focus on Lawson exponential methods. They were deduced in [13] by considering a change of variables which converts the stiff system into a non-stiff one to which a Runge–Kutta method is applied. Lawson method is then obtained by undoing the corresponding change of variables. In spite of the fact that this special type of exponential methods is one of the oldest, a thorough error analysis for it has not been performed in the literature until now because it does not satisfy some simplifying conditions which lead directly to (at least) stiff order 1 in explicit exponential methods [10]. Those conditions imply the conservation of equilibria of autonomous problems but the fact that they are not satisfied does not mean that they are not valuable methods, as it has already been proved in the literature [6–8]. The aim of this manuscript is to perform an analysis of the error when integrating linear problems, for which these methods can be seen as exponential quadrature rules. We will justify when order reduction is shown and when it is not. That will depend on regularity of the solution and conditions of annihilation or periodicity at the boundary. Moreover, when there is order reduction, we will explain here the precise order which is observed and, in another paper [5], we give a technique to avoid it. As this technique is very cheap, Lawson methods become a very valuable tool to integrate linear initial boundary value problems. This is due to the fact that they are able to integrate them accurately in an explicit and stable way without requiring any conditions on the coefficients of the method which could increase the computational cost, as is the case with other exponential Runge–Kutta methods in the literature until now [10,14].

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The analysis will be performed with an abstract formulation of the problem, under the general framework of C_0-semigroups in Banach spaces. More precisely, we will consider the well-posed linear abstract initial value problem

$$\begin{aligned} u'(t) &= Lu(t) + f(t), \quad 0 < t < T, \\ u(0) &= u_0, \end{aligned} \tag{1}$$

where u_0 and $f(t)$ ($t \in [0, T]$) belong to a Banach space X and $L : D(L) \subset X \to X$ is a linear operator which is the infinitesimal generator of a C_0-semigroup $\{e^{tL}\}_{t\geq 0}$; so that, for certain constants $M > 0$, $\omega \in \mathbb{R}$, we have

$$\|e^{tL}\| \leq Me^{\omega t}, \quad 0 < t < T. \tag{2}$$

For the sake of simplicity, we suppose that $\omega < 0$. As a consequence, the operator L is invertible and L^{-1} is bounded. For $\nu \geq 0$, we denote $X_\nu = D((-L)^\nu)$, endowed with the norm $\|x\|_\nu = \|(-L)^\nu x\|$, for each $x \in X_\nu$. We remark that the operators $(-L)^{-\nu}$ are bounded [21]. The use of these powers is useful to prove fractional orders of convergence. To obtain this fractional order, and to use a summation-by-parts argument for the global error, we need additional assumptions on the operator L. As for boundary conditions, homogeneous Dirichlet, Neumann and Robin conditions can be considered and they will be implicitly satisfied by all functions in $D(L)$. As we will see in Section 3, time regularity of the solution is not sufficient to obtain the classical order of the numerical integration; the order observed depends on the fact that u and some of its time derivatives belong to the domain of a certain power of L and therefore that means more conditions of annihilation at the boundary which are not natural for the solution of (1).

The case of a pure initial value problem or periodic boundary conditions can also be studied under this framework. Notice that here belonging to the domain of a certain power of L just means more regularity in space but no additional artificial conditions at the boundary. Therefore, for initial periodic boundary value problems, no order reduction is shown if the solution is regular enough in space and time.

The paper is structured as follows. In Section 2, we study sufficient conditions on the data of the problem (1) (u_0 and f) in order to assure a certain regularity of the solution u. In Section 3, we offer a thorough analysis of the local and global errors which are observed under certain assumptions on that exact solution. Finally, in Section 4, we corroborate these results when we apply Lawson methods to integrate in time some problems in 1 and 2 dimensions.

2. On the regularity of the solution

In this section, we firstly state Lemma 2.1 and Proposition 2.2, whose proof can be easily deduced from Theorem 2.4 and Corollary 2.5 in Section 4.1 of [19].

Lemma 2.1. *Let L be the infinitesimal generator of a C_0 semigroup e^{tL}. If $f \in C^1([0, T], X)$, then $\int_0^t e^{sL} f(t-s)ds \in D(L)$ and*

$$L\int_0^t e^{sL} f(t-s)ds = e^{tL} f(0) - f(t) + \int_0^t e^{sL} f'(t-s)ds.$$

Proposition 2.2. *Let L be the infinitesimal generator of a C_0 semigroup e^{tL}. If $u_0 \in D(L)$ and $f \in C^1([0, T], X)$, the unique solution u of the initial value problem (1) belongs to $C^1([0, T], X) \cap C([0, T], D(L))$ and is given by*

$$u(t) = e^{tL}u_0 + \int_0^t e^{sL} f(t-s)ds. \tag{3}$$

Besides, its derivative is given by

$$u'(t) = e^{tL}Lu_0 + e^{tL} f(0) + \int_0^t e^{sL} f'(t-s)ds. \tag{4}$$

Now, we can obtain more regularity with a recursive argument.

Theorem 2.3. *Let $q \geq 1$ be an integer number and let L be the infinitesimal generator of a C_0 semigroup e^{tL}.*

Assume that $f \in C^q([0, T], X)$, $v_0 = u_0 \in D(L)$ and, for $1 \leq j \leq q-1$,

$$v_j = Lv_{j-1} + f^{(j-1)}(0) \in D(L). \tag{5}$$

Then, the unique solution of the initial value problem (1) belongs to $C^q([0, T], X) \cap C([0, T], D(L))$ and is given by