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# An efficient two-level finite element algorithm for the natural convection equations ☆

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ABSTRACT

An efficient two-level finite element algorithm for solving the natural convection equations is developed and studied in this paper. By solving one small nonlinear system on a coarse mesh $H$ and two large linearized problems on a fine mesh $h = O(H^{\frac{7-\varepsilon}{2}})$ with different loads, we can obtain an approximation solution $(u^h, p^h, T^h)$ with the convergence rate of same order as the usual finite element solution, which involves one large nonlinear natural convection system on the same fine mesh $h$. Furthermore, compared with the results of Si's algorithm in 2011, the given algorithm costs less computed time to get almost the same precision.

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## 1. Introduction

In this article, a two-level method for solving a nonlinear system arising from finite element discretizations of the equilibrium natural convection equations is developed and studied. Let $\Omega$ be a bounded open domain in $\mathbb{R}^2$ with disjoint domains $\Omega_s$ and $\Omega_f$, assumed to have a Lipschitz continuous boundary $\partial\Omega$. Suppose $\Gamma_T = \partial\Omega \backslash \Gamma_B$, where $\Gamma_B$ is a regular open subset of $\partial\Omega$. Consider the following stationary natural convection equations including solid media in dimensionless form [2,3,5]

$$\begin{aligned} -Pr\Delta u + (u\cdot\nabla)u + \nabla p &= Pr\,Ra\,jT \quad \text{in } \Omega_f, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega_f, \\ -\nabla\cdot(\kappa\nabla T) + (u\cdot\nabla)T &= \gamma \quad \text{in } \Omega, \\ u = 0 \ \text{on } \partial\Omega_f, &\quad u \equiv 0 \ \text{in } \Omega - \Omega_f = \Omega_s, \\ T = 0 \ \text{on } \Gamma_T, &\quad \frac{\partial T}{\partial n} = 0 \ \text{on } \Gamma_B, \end{aligned} \tag{1}$$

where $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $T(x)$ the temperature, $\gamma$ the forcing function, $Pr, Ra > 0$ the Prandtl and Rayleigh numbers, $\kappa > 0$ the thermal conductivity parameter, $j = (0, 1)$ the two-dimensional vector, $n$ the outward unit normal to the $\Gamma_B$ and the symbols $\Delta$, $\nabla$ and div denote the Laplacian, gradient and divergence

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operators, respectively. Moreover, take the case $\kappa \equiv \kappa_f$ in $\Omega_f$ and $\kappa \equiv \kappa_s$ in $\Omega_s$, where $\kappa_f$ and $\kappa_s$ are positive constants denoted the thermal conductivity for the different media.

The natural convection problem (1), which does not only incorporate the velocity vector field as well as the pressure field but also contains the temperature field, plays an important role in many current scientific, engineering, and industrial applications. This problem constitutes an important system of equations in atmospheric dynamics atmospheric fronts, katabatic winds, natural ventilation, solar collectors, dense gas dispersion, cooling of electronic equipments and nuclear reactors, and insulation with double pane window. Therefore, much effort has been throwing to the development of efficient numerical methods for investigating this problem. At the time of writing, there are numerous works devoted to the development of efficient methods for the natural convection problem ([1,16,18,24,27,29,33] and the references therein). In addition, an explicit finite element algorithm for convection heat transfer problems has been presented by Manzari [28]. He has used a standard Galerkin finite element method for spatial discretization and an explicit multistage Runge–Kutta scheme to march in the time domain. In 2011, by using the two-level strategy [34,35], Si et al. [30] have presented two-level Newton iterative mixed finite element methods for solving the stationary conduction–convection problem. This method involves solving a linearized conduction–convection problem on the coarse mesh $H$ and a simply linearized conduction–convection problem on the fine mesh $h = O(H^{\frac{3}{2}})$ based on the Taylor–Hood element, which can save a large amount of computed time. Some details of two-level method can be found in the works of Layton et al. [21–23], Ervin et al. [8], Girault and Lions [9], He et al. [12–15], Li [25] and Huang et al. [17,19,20]. Recently, Dai and Cheng [6] have considered a two-grid method based on Newton iteration for the Navier–Stokes equations, which has a great improvement both on errors in $H^1$ and $L^2$-norms.

In this article, an efficient two-level finite element algorithm is developed and studied for the natural convection equations. This method includes three steps: solve one small nonlinear natural convection system on a coarse mesh with mesh size $H$, and a large linearized problem on a fine mesh with mesh size $h = O(H^{\frac{7-\varepsilon}{2}})$ based on Newton iteration and at last still solve the same linear problem with different loads on the same fine mesh. Hence, it can save a large amount of computational time due to that, only two linear problems with the same stiffness matrix need to be solved on the fine grid. Similar two-level finite element method for solving conduction–convection equations [30] and for the Navier–Stokes equations [6] was studied. Thus, this paper can be cast in the framework of Si et al. [30] and Dai and Cheng [6]. However, there are essential differences between the presented two-level method and that studied in [30]. A correction problem on fine mesh is solved at the third step and the scaling between the coarse mesh size $H$ and the fine mesh size $h$ becomes $h = O(H^{\frac{7-\varepsilon}{2}})$. While compared to the usual two-level method for the considered problem [30], the presented method can save more computed time because of the improved scaling. Hence, this paper can be considered as a complement of the work of Si et al.

The rest of the article is organized as follows: In the next section, we introduce some notation and well-known results used throughout this paper. Subsequently, a finite element method for the considered equations is recalled in Section 3. Then, in Section 4, an efficient two-level finite element algorithm for the natural convection equations is defined, and stability and optimal order estimates of the algorithm are obtained. Moreover, numerical examples are shown to verify the good properties of the shown method in Section 5. Finally, we end with a short conclusion.

## 2. Preliminaries

For the mathematical setting of problem (1), we introduce the following Hilbert spaces [5]:

$$X = H_0^1(\Omega_f)^2, \quad W = \{s \in H^1(\Omega) : s = 0 \text{ on } \Gamma_B\},$$

$$M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

The space $L^2(\Omega)$ is equipped with the $L^2$-scalar product $(\cdot,\cdot)$ and $L^2$-norm $\|\cdot\|_0$. The spaces $X$ and $W$ are endowed with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|\nabla u\|_0$. Standard definitions are used for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $\|\cdot\|_{m,p}$, $m, p \geq 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. Besides, define the dual space of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$, with its norm $\|f\|_{-1} = \sup_{v \in H_0^1(\Omega)} \frac{|(f,v)|}{\|\nabla v\|_0}$. In addition, it is well known [10] that

$$\|v\|_0 \leq c_1 \|\nabla v\|_0, \quad \forall v \in X (\text{or } W), \tag{2}$$

where $c$ (with or without a subscript) will denote a generic positive constant, which is independent of the mesh size, but may depend on $\Omega$ and other parameters introduced in this paper.

We define two continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u, v) = (\nabla u, \nabla v), \quad \forall u, v \in X,$$

$$d(v, q) = (q, \operatorname{div} v), \quad \forall v \in X, \ \forall q \in M,$$

and a trilinear form on $X \times X \times X$ by

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