Applied Numerical Mathematics 118 (2017) 117–130

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Optimized interface conditions in domain decomposition methods to solve reaction–diffusion problems with strong heterogeneity in the coefficients in a sectorial domain

Chokri Chniti [a,b,c]

[a] Umm Al-Qura University, College of Education Al-Qunfudah, Mathematics Department, Macca, Kingdom of Saudi Arabia
[b] University of Carthage, Preparatory Institute for Engineering Studies of Nabeul, Mrezgua 8000 Nabeul, Tunisia
[c] University of Tunis El Manar, Faculty of Sciences of Tunis, Department of Mathematics, Campus University, 2092 Tunis, Tunisia

ARTICLE INFO

Article history:
Received 21 August 2015
Received in revised form 23 January 2017
Accepted 26 February 2017
Available online 7 March 2017

Keywords:
Kondratiev theory
Corner singularity
Domain decomposition methods
Transmission conditions
Schwarz algorithm

ABSTRACT

The aim of this paper is to derive appropriate second order transmission boundary conditions near the corner used in domain decomposition methods to study the reaction–diffusion problems ("$-\nabla.(\nu(x)\nabla.) + \eta(x)$.") with strong heterogeneity in the coefficients in a singular non-convex domain with Neumann and Dirichlet boundary condition. These transmission conditions will be tested and compared numerically with other approaches.

1. Introduction

Domain decomposition methods are now well understood in the case of a regular domain decomposed into regular subdomains, and then solve the subdomain problems. Interface conditions are crucial in domain decomposition methods and their design has been the subject of many works. We must impose boundary conditions along the interfaces between subdomains which were not part of the boundary of the whole domain. An astute choice of artificial boundary conditions can accelerate the convergence of the algorithm used. In [15], Robin interface conditions are proposed and other model problem with different interface boundary condition was proposed in [1,7,9–11,18]. The classical Schwarz method is based on Dirichlet boundary conditions. Overlapping subdomains are necessary to ensure convergence. It has been proposed independently in [12,15] to use more general interface conditions in order to accelerate the convergence and to allow for non-overlapping decomposition. They are optimal in terms of iteration counts [19]. A good comprehension of the singular case is a challenge in my research, for example problems with corners. In [4–6] the authors present a local improvement of domain decomposition methods for the model operator $-\Delta$ for which either the geometry or the domain decomposition presents conical singularities. In [2], the author presents an improvement near the corner for second order elliptic operators $L_1 = -\nabla.(\nu(x)\nabla.)$ in a non-convex domain with Dirichlet boundary condition and exactly a comparison between the optimized transmission conditions derived from the regular cases (see [8]) and the modified transmission conditions close the corners obtained to show the improvement. In [3], Chniti certifies that the same approach is appropriate for the reaction–diffusion problems ($L_2 = -\nabla.(\nu(x)\nabla.) + \eta(x)$) with Dirichlet boundary condition in the case when the domain

E-mail address: cachniti@uqu.edu.sa.

http://dx.doi.org/10.1016/j.apnum.2017.02.015

decomposition presents conical singularities and not geometry singularities. The typical equation we consider in this paper is the reaction–diffusion problem with highly heterogeneous coefficients

$$\begin{cases} -\nabla.(\nu(x)\ \nabla u) + \eta(x)u = f \ \text{ in } \Omega \subset \mathbb{R}^2 \\ \quad \Lambda_1 u \ = 0 \text{ on } \gamma, \\ \quad \Lambda_2 u \ = g \text{ on } \partial\Omega \setminus \gamma \end{cases} \tag{1}$$

where $\gamma \subset \partial\Omega$ corresponds to the piece of the boundary and the global domain Ω is polygonal and possibly non-convex, the boundary operator $\Lambda_i, i \in \{1, 2\}$ appearing in the above relations denotes either the trace or the normal trace operator on $\partial\Omega$ and f, g are the right-hand side. The scalar diffusion coefficients $\nu(x)$ and $\eta(x)$ are piecewise constant functions:

$$\nu(x) = \begin{cases} \nu_1 \text{ in } \Omega_1 \\ \nu_2 \text{ in } \Omega_2 \end{cases} \qquad \eta(x) = \begin{cases} \eta_1 \text{ in } \Omega_1 \\ \eta_2 \text{ in } \Omega_2 \end{cases}$$

Here, Ω_1 and Ω_2 form a natural decomposition of the domain Ω into nonoverlapping subdomains. The question arises: Why L_2 is considered in this paper and not L_1. Some reason allows us to consider L_2. First, the two operators were studied in details in the case of Domain Decomposition Methods with regular interface between subdomains and the optimized coefficient involved in the transmission condition of second order is derived, see [8], this helps us to make a comparison between our approach of optimization near the corner and the other ones (without corner) and this is the aim of this paper in order to certify that the idea of optimization near the corner gives an improvement. Second, we will test our strategy with Neumann boundary condition and a precise modification of the interface condition near the corner will be presented in order to make the numerical and efficient tests. Third, there is a problem if we use L_1 with Neumann boundary condition, in fact the solution (u) is defined just by derivative, so this defined too a constant (if u is a solution then $u + 1$ is also a solution) then the solution is not unique and linear algebra tells us that the right hand side must be orthogonal to the kernel of the operator. Here the problem is defined to a constant. We write the variational or weak form of the problem using the operator L_1 is generally well posed if we do not have only Neumann boundary condition on the whole boundary of the global domain, and one way to avoid this problem one can get a compatibility condition by adding to the operator a constant η. We study in this paper the influence of the transmission conditions on the Schwarz algorithm [10] for reaction–diffusion problems with Dirichlet and Neumann boundary condition on the whole of the boundary of the global domain. We numerically test improved transmission conditions with second-order tangential derivatives, which were derived from an asymptotic analysis of the Schwarz algorithm near the corners of the domain with different boundary condition and exactly we will consider two cases: Dirichlet problem and Neumann problem. We consider the Neumann problem, because it does not satisfy the condition of isomorphism in weighted Sobolev spaces which is used in [16,17]. The theoretical optimality of the asymptotic analysis based on the matching of the main singularities within Kondratiev theory [14] is confirmed by numerical computations.

The outline of the paper is the following: In Section 2, we introduce the model problem and the interface boundary condition used in this paper. In Section 3 a rapid review of some consequences of Kondratiev's theory [14] is presented. In Section 4, we present our strategy for improving the convergence rates around corner singularities. In Section 5, we derive an optimal choice of the transmission conditions near the corner. In all the rest, we give some practical examples and numerical experiments which confirm the optimality of such coefficients and we conclude.

2. Interface boundary conditions

First, let us recall that the aim of the analysis is to improve the behavior of domain decomposition algorithms locally around the corner of sub-domains. The problem amounts to determining the coefficients of interface boundary conditions so that the domain decomposition algorithm has rapid convergence "in terms of iterations". Specific problems occur in the presence of conical singularities. Starting from the usual method used for regular interfaces, we derive a local improvement of the interface boundary condition near the corner. A brief reminder of the transmission condition of the regular case will be proposed and an adaptation of the transmission conditions of the regular case is necessary near the corner in order to avoid the Dirichlet interface conditions which do not transmit well the information from one sub-domain to its neighboring ones, see [5] for more details.

2.1. Case of a regular interface

The reaction–diffusion problem considered in our study is:

$$\begin{cases} -\nabla.(\nu(x)\nabla u) + \eta(x)u \quad = f \ \text{ for } \quad x \in \Omega \subset \mathbb{R}^2 \\ \qquad\qquad\qquad |u| \ < +\infty \text{ as } \quad x \to \infty \end{cases} \tag{2}$$

where f is the right-hand side and the scalar diffusion coefficients $\nu(x)$ and $\eta(x)$ are the piecewise constant functions

$$\nu(x) = \begin{cases} \nu_1 \text{ in } \Omega_1 \\ \nu_2 \text{ in } \Omega_2 \end{cases} \qquad \eta(x) = \begin{cases} \eta_1 \text{ in } \Omega_1 \\ \eta_2 \text{ in } \Omega_2 \end{cases}$$