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BDF-type shifted Chebyshev approximation scheme for fractional functional differential equations with delay and its error analysis

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A B S T R A C T

A numerical method for fractional order differential equations (FDEs) and constant or time-varying delayed fractional differential equations (FDDEs) is constructed. This method is of **BDF**-type which is based on the interval approximation of the true solution by truncated shifted Chebyshev series. This approach can be reformulated in an equivalent way as a Runge–Kutta method and its Butcher tableau is given. A detailed local and global truncating errors analysis is deduced for the numerical solutions of FDEs and FDDEs. Illustrative examples are included to demonstrate the validity and applicability of the proposed approach.

1. Introduction

This paper is devoted to present a new **BDF**-type method based on approximations by truncated shifted Chebyshev series, appropriate for Initial value problems of FDEs and FDDEs.

$$D^{(\beta)}y(t) = f(t, y(t), y(t-\tau)), \quad t \in [0, L], \quad 0 < \beta \le 1, \tag{1}$$

with initial condition

$$y(t) = \phi(t), \quad t \in [-\tau, 0], \tag{2}$$

such that $\tau > 0$ is the delay term which may be constant or variable with respect to t. The Caputo fractional derivative operator $D^{(\beta)}$ of order β is defined in the following form [12]:

$$D^{(\beta)}f(x) = \frac{1}{\Gamma(m-\beta)} \int\limits_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\nu-m+1}} d\xi, \qquad \nu > 0,$$

where $m-1 < \beta \le m$, $m \in \mathbb{N}$, $x > 0$.

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It is worth mentioning that FDDEs (1)–(2) will be reduced to FDEs when $\tau = 0$. Fractional differential equations have been an active area of research owing to the intensive development of the theory of fractional calculus itself and their applications in various fields of science, such as physics [23], chemistry [13], engineering [21,26], automatic control [27,7,18], etc. The insertion of a time delay on the right side of the fractional differential equation balances the equation by also adding a degree of system memory on the right side of the equation. By including both fractional derivatives and finite time delays in some models (with memory and heritage properties) [4], more complete and realistic models can be established. Machado [28] proposed the calculation of fractional algorithms based on time-delay systems. The study analyzed the memory properties of fractional operators and their relation with time delay. The effect of delay on the chaotic behavior of fractional order Liu system has been investigated in [3]. A delayed fractional order financial system was proposed and the complex dynamical behaviors of such a system are discussed by numerical simulations [29]. In the past few years, many scholars studied the existence problems of solutions for fractional differential equations. Existence, uniqueness, and structural stability of solutions of nonlinear differential equations of fractional order were studied by Diethelm and Ford [12]. Existence and uniqueness theorems for the initial value problem for the system of fractional differential equations were proposed by Gejji and Babakhani in [9]. Analysis of solutions for FDDEs has been carried in some approaches since time delay is unavoidable in the real systems. By the Banach fixed point theorem and the nonlinear alternative of Leray–Schauder type, Benchohra et al. [2] first discussed the existence of solutions for initial value problems of fractional order $0 < \alpha < 1$ with infinite delay and Riemann–Liouville fractional derivative. They also discussed operator type neutral equations in the same way. Deng and Qu [11] discussed the same problem in [2] and obtained some new uniqueness results. By fixed point theorem, Zhou et al. [32] considered the initial value problem of fractional neutral differential equations with infinite delay and Caputo fractional derivative. Agarwal et al. [1] considered the similar problems of Caputo fractional neutral equations with bounded delay. More recently, Yang et al. [31] gave a global existence and uniqueness of initial value problems for nonlinear higher fractional equations with delay by fixed point theory. By fixed point theory the nonlinear alternative of Leray–Schauder type, and the properties of absolutely continuous functions space, the authors obtained some new results involving local and global solutions.

Numerical methods for solving such problems which contain both fractional derivatives and time delays are considerably less developed. In this regard, Wang [30] lately approximated the delayed fractional order differential equation by combining the general Adams–Bashforth–Moulton method with the linear interpolation method. Morgado et al. [22] studied numerically a special form of initial value problem for a linear fractional differential equation with finite delay by adaptation of a fractional backward difference method. A new predictor–corrector method had been developed to solve fractional delay differential equations [10]. Related error analysis of the method had been presented and the error bounds also obtained. Bahrawy proposed an accurate and robust approach to approximate the solution of functional Dirichlet boundary value problem with a type of variable order Caputo fractional derivative [5]. A method which is a generalization from finite difference method, has been provided to solve a class of fractional delay differential equations [20].

In this approach, we present a **BDF** difference scheme based on approximations of Clenshaw and Curtis type [8]. We do not use an approximation of the function f in the right hand side of the fractional differential equation but we use a truncated Chebyshev series instead. This kind of series results to be the interpolating polynomial that passes through certain intermediate points known as interpolation points and impose as in the **BDF** methods that the derivatives at these points coincide with the values of the function f. The structure of this paper is arranged as follows: we present a definition of shifted Chebyshev polynomials and their analytic formula in the next section. In the third section, a derivation of the method is obtained. The proposed method is represented as one step recurrence formula, local truncating error and global truncating error are clarified in the fourth section. The fifth section is devoted to introduce some numerical examples to ensure the accuracy of the presented method. Finally, paper ends with a brief conclusion.

2. Shifted Chebyshev polynomials

The shifted Chebyshev polynomials $T_n^*(x)$ of degree n which are defined on the interval $[0, L]$ such that $T_0^*(x) = 1$, $T_1^*(x) = \frac{2x}{L} - 1$, have the following analytic form [19]:

$$T_n^*(x) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k-1)! 2^{2k}}{(n-k)!(2k)! L^k} x^k, \tag{3}$$

where, $T_n^*(0) = (-1)^n$, $T_n^*(L) = 1$. The orthogonality condition of these polynomials is

$$\int_0^L T_j^*(x) T_k^*(x) w(x) dx = \delta_{jk} h_k, \tag{4}$$

where, the weight function $w(x) = \frac{1}{\sqrt{Lx - x^2}}$, $h_k = \frac{b_k}{2}\pi$, with $b_0 = 2$, $b_k = 1$, $k \geq 1$.

The function $y(x)$ which belongs to the space of square integrable in $[0, L]$, may be expressed in terms of shifted Chebyshev polynomials as