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Quadrature rules and asymptotic expansions for two classes of oscillatory Bessel integrals with singularities of algebraic or logarithmic type ☆

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ABSTRACT

In this paper we mainly focus on the quadrature rules and asymptotic expansions for two classes of highly oscillatory Bessel integrals with algebraic or logarithmic singularities. Firstly, by two transformations, we transfer them into the standard types on $[-1, 1]$, and derive two useful asymptotic expansions in inverse powers of the frequency ω. Then, based on the two asymptotic expansions, two methods are presented, respectively. One is the so-called Filon-type method. The other is the more efficient Clenshaw-Curtis–Filon-type method, which can be implemented in $O(N \log N)$ operations, based on Fast Fourier Transform (FFT) and fast computation of the modified moments. Here, through large amount of calculation and analysis, we can construct two important recurrence relations for computing the modified moments accurately, based on the Bessel's equation and some properties of the Chebyshev polynomials. In particular, we also provide error analysis for these methods in inverse powers of the frequency ω. Furthermore, we prove directly from the presented error bounds that these methods share the advantageous property, that the larger the values of the frequency ω, the higher the accuracy. The efficiency and accuracy of the proposed methods are illustrated by numerical examples.

1. Introduction

The useful oscillatory integrals, such as Bessel transforms, have been extensively investigated and applied in mathematical and numerical modeling of oscillatory phenomena in many areas of sciences and engineering such as astronomy, electromagnetics, acoustics, scattering problems, physical optics, electrodynamics, and applied mathematics [2–5,13,22]. In this paper, we are concerned with the quadrature rules and asymptotic expansions for singular oscillatory Bessel transforms of the forms

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$$I_1[f]=\int_0^1 x^{\alpha} f(x) J_m(\omega/x^{\beta})dx, \tag{1.1}$$

$$I_2[f]=\int_0^1 x^{\alpha} \ln(x) f(x) J_m(\omega/x^{\beta})dx, \tag{1.2}$$

where $\omega, \beta > 0, \alpha+\beta > -1$ are real numbers, f is a non-oscillatory, sufficiently smooth function on $[0,1]$ and $J_m(z)$ is the Bessel function [1] of the first kind and of order m with $Re(m) > -1$. Particularly, it should be noticed that transforms (1.1) and (1.2) are integrals with singularities of algebraic or logarithmic type, and oscillatory Bessel kernel functions, respectively. Moreover, the two classes of integrals (1.1) and (1.2), often arise in the numerical approximations of solutions to Volterra integral equations of the first kind involving highly oscillatory kernels with algebraic and logarithmic singularities [4,5]. Further, they can be taken as model integrals appearing in boundary integral equations for high-frequency acoustic scattering (e.g., high-frequency Helmholtz equation in two dimensions), where the kernels have algebraic or logarithmic singularities on the diagonal, which is also our main target application. Therefore, it is of great importance for the study of the numerical integration of such integrals.

In most of the cases, such integrals cannot be calculated analytically and one has to resort to numerical methods. The numerical evaluation can be difficult when the parameter ω is large, because in that case the integrand is highly oscillatory. The singularities of algebraic or logarithmic type and possible high oscillations of the integrands in (1.1) and (1.2) make the above integrals very difficult to approximate accurately using standard methods, e.g., *Gaussian quadrature rules*. It is well known that a prohibitively large number of quadrature points is needed if one uses a classic rule such as Gaussian quadratures, or any quadrature method based on (piecewise) polynomial interpolation of the integrands.

In the last few decades, much progress has been made in developing numerical schemes for generalized Bessel transform $\int_a^b f(x) J_m(\omega g(x))dx$ without singularity (see [10,11,6–8,16,28,34,37,42–44]). In addition, the orthogonal polynomial expansion method [29], the *Filon-type method* [9], and the *Clenshaw-Curtis–Filon method* [9,47] were given for approximating the Bessel transform $\int_0^1 x^{\alpha}(1-x)^{\beta} f(x) J_m(\omega x)dx, \alpha, \beta > -1$, with singularities at the two endpoints, respectively. Xu and Xiang in [45,46] proposed the *Clenshaw-Curtis–Filon method* for computing the oscillatory Airy integrals $\int_0^1 x^{\alpha}(1-x)^{\beta} f(x) Ai(-\omega x)dx, \alpha, \beta > -1$, with singularities at the two endpoints, and the highly oscillatory finite Hankel transform $\int_0^1 f(x) H_{\nu}^{(1)}(\omega x)dx$, respectively. The first author of this paper and Ling in [25] also presented the *Clenshaw-Curtis–Filon method* for computing many integrals including different singular oscillatory kernel functions.

Here, we would also like to mention several other related papers [17,20,21,26,27], where they presented numerical methods for computing the singular oscillatory integrals of the type

$$\int_0^1 x^{\alpha} f(x) e^{i\omega/x^{\beta}} dx, \tag{1.3}$$

where $\omega, \beta > 0, \alpha+\beta > -1$. Gautschi in [17] considered the computation of (1.3) by using the modified Chebyshev algorithm for the case $\alpha = 0$, $\beta = 1$ and $\omega = 1$. In [20,21], based on the work of Gautschi, for the forms of α, β, ω in a certain range, such as $\alpha = 0$, $0 < \omega \le 100$ and $0.00001 \le \beta \le 10000$, Hascelik constructed a suitable *Gauss quadrature rule* by using the modified Chebyshev algorithm to compute the integrals (1.3). Unfortunately, the methods in [17,20,21] required the use of high precision arithmetic and the complexity of the modified Chebyshev algorithm in terms of arithmetic operations is $O(n^2)$ for a n-point Gauss rule [18]. Moreover, the proposed Gauss quadrature rules [20,21] were unstable for the case $\omega \gg 1$. Meanwhile, Hascelik [21] gave appropriate Filon-type methods for (1.3), with related error bounds. Recently, the authors of [26] extended the *numerical steepest descent method* to the computation of highly oscillatory integrals of the form (1.3) when $\alpha = 0$. In fact, the numerical steepest descent method can also be applied to (1.3) when $\alpha+\beta > -1$, if f is analytical in the region $G_1 = \{z \in \mathbb{C} \mid |z| \le 1\}$ containing $[0,1]$. In order to relax the strict requirement that f is analytic in G_1, more general methods presented in [27] were available for just sufficiently smooth f on $[0,1]$.

Although the considered integrals (1.1) and (1.2) in this paper are similar to (1.3), the integrands in (1.1) and (1.2) contain a logarithmic singularity and more complicated kernel functions (Bessel functions), which makes the above integrals (1.1) and (1.2) more difficult to approximate accurately. And, to the best of our knowledge, so far little research has been done on the numerical computation of the integrals (1.1) and (1.2) with algebraic or logarithmic singularities.

Consequently, our aim is to design quadrature rules and asymptotic expansions for such integrals (1.1) and (1.2). In the next section, we derive two key asymptotic expansions in inverse powers of ω. Then, based on the asymptotic expansions, a Filon-type method and its error analysis are given. Section 3 presents a Clenshaw-Curtis–Filon-type method and its error analysis for computing the integrals (1.1) and (1.2). Here, the required modified moments can be accurately calculated by constructing two important recurrence relations. Moreover, in the sections 2–3, we also provide some numerical examples to show the accuracy and efficiency of these quadrature rules. All these presented methods share an advantageous property that the accuracy greatly improves when ω increases.