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Shift techniques for Quasi-Birth and Death processes: Canonical factorizations and matrix equations [☆]

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ABSTRACT

We revisit the shift technique applied to Quasi-Birth and Death (QBD) processes He et al. (2001) [13] in functional form by bringing the attention to the existence and properties of canonical factorizations. To this regard, we prove new results concerning the solutions of the quadratic matrix equations associated with the QBD. These results find applications to the solution of the Poisson equation for QBDs.

1. Introduction

Quadratic matrix equations of the kind

$$A_{-1} + (A_0 - I)X + A_1 X^2 = 0, \tag{1}$$

where A_{-1}, A_0, A_1 are given $n \times n$ matrices, are encountered in many applications, say in the solution of the quadratic eigenvalue problem, like vibration analysis, electric circuits, control theory and more [21,15]. In the area of Markov chains, an important application concerns the solution of Quasi-Birth-and-Death (QBD) stochastic processes, where it is assumed that A_{-1}, A_0 and A_1 are nonnegative matrices such that $A_{-1} + A_0 + A_1$ is stochastic and irreducible [18,3].

For this class of problems, together with (1), the dual equation $X^2 A_{-1} + X(A_0 - I) + A_1 = 0$ has a relevant interest. It is well known that both (1) and the dual equation have minimal nonnegative solutions G and R, respectively, according to the component-wise ordering. More specifically, $G \geq 0$, means that all the entries of G are nonnegative, moreover, G is a minimal nonnegative solution if $G \geq 0$ and for any other nonnegative solution Y it holds that $Y - G \geq 0$. The solutions G and R, which can be explicitly related to one another [18,20], have an interesting probabilistic interpretation and their computation is a fundamental task in the analysis of QBD processes. Moreover they provide the factorization $\varphi(z) = (I - zR)K(I - z^{-1}G)$ of the Laurent polynomial $\varphi(z) = z^{-1}A_{-1} + A_0 - I + zA_1$, where K is a nonsingular matrix. A factorization of this kind is canonical if $\rho(R) < 1$ and $\rho(G) < 1$, where ρ denotes the spectral radius. It is said to be weak canonical if $\rho(R) \leq 1$ and $\rho(G) \leq 1$.

We introduce the matrix polynomial $B(z) = A_{-1} + z(A_0 - I) + z^2 A_1 = z\varphi(z)$ and define the roots of $B(z)$ as the zeros of the polynomial $\det B(z)$. If ξ is a root of $B(z)$ we say that v is an eigenvector associated with ξ if $v \neq 0$ and $B(\xi)v = 0$. The location of the roots of $B(z)$ determines the classification of the QBD as positive, null recurrent or transient, and governs the convergence and the efficiency of the available numerical algorithms for approximating G and R [3]. In particular, $B(z)$ has

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always a root on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, namely, the root $\xi = 1$, and the corresponding eigenvector is the vector e of all ones, i.e., $B(1)e = 0$.

If the QBD is recurrent, the root $\xi = 1$ is the eigenvalue of largest modulus of the matrix G and $Ge = e$. In the transient case, that root is the eigenvalue of largest modulus of R. These facts have been used to improve convergence properties of numerical methods for computing the matrix G. The idea, introduced in [13] and based on the results of [6], is to "shift" the root $\xi = 1$ of $B(z)$ to zero or to infinity, and to construct a new quadratic matrix polynomial $B_{\boldsymbol{s}}(z) = A^{\boldsymbol{s}}_{-1} + z(A^{\boldsymbol{s}}_0 - I) + z^2 A^{\boldsymbol{s}}_1$ having the same roots as $B(z)$, except for the root equal to 1, which is replaced with 0 or infinity. Here the super-(sub-)script $\boldsymbol{s}$ means "shifted". This idea has been subsequently developed and applied in [5,7,10–12,16,17,19].

In this paper we revisit the shift technique in functional form, and we focus on the properties of the canonical factorizations. In particular, we prove new results concerning the existence and properties of the solutions of the quadratic matrix equations obtained after the shift.

By following [3], we recall that in the positive recurrent case the root $\xi = 1$ can be shifted to zero by multiplying $B(z)$ to the right by a suitable function (right shift), while in the transient case the root $\xi = 1$ can be shifted to infinity by multiplying $B(z)$ to the left by another suitable function (left shift). In the null recurrent case, where $\xi = 1$ is a root of multiplicity 2, the shift operation can be applied both to the left and to the right so that the two roots which coincide with 1 are shifted to zero and to infinity, respectively (double shift). In all the cases, the new Laurent matrix polynomial $\varphi_{\boldsymbol{s}}(z) = z^{-1} B_{\boldsymbol{s}}(z)$ is invertible on an annulus containing the unit circle in the complex plane and we prove that it admits a canonical factorization which is related to the weak canonical factorization of $\varphi(z)$. As a consequence, we relate G and R with the solutions $G_{\boldsymbol{s}}$ and $R_{\boldsymbol{s}}$ of minimal spectral radius of the matrix equations $A^{\boldsymbol{s}}_{-1} + (A^{\boldsymbol{s}}_0 - I)X + A^{\boldsymbol{s}}_1 X^2 = 0$ and $X^2 A^{\boldsymbol{s}}_{-1} + X(A^{\boldsymbol{s}}_0 - I) + A^{\boldsymbol{s}}_1 = 0$, respectively.

A less trivial issue is the existence of the canonical factorization of $\varphi_{\boldsymbol{s}}(z^{-1})$. We show that such factorization exists and we provide an explicit expression for it, for the three different kinds of shifts. The existence of such factorization allows us to express the minimal nonnegative solutions $\widehat{G}$ and $\widehat{R}$ of the matrix equations $A_{-1}X^2 + (A_0 - I)X + A_1 = 0$ and $A_{-1} + X(A_0 - I) + X^2 A_1 = 0$, in terms of the solutions of minimal spectral radius $\widehat{G}_{\boldsymbol{s}}$ and $\widehat{R}_{\boldsymbol{s}}$ of the equations $A^{\boldsymbol{s}}_{-1}X^2 + (A^{\boldsymbol{s}}_0 - I)X + A^{\boldsymbol{s}}_1 = 0$ and $A^{\boldsymbol{s}}_{-1} + X(A^{\boldsymbol{s}}_0 - I) + X^2 A^{\boldsymbol{s}}_1 = 0$, respectively.

The existence of the canonical factorizations of $\varphi_{\boldsymbol{s}}(z)$ and $\varphi_{\boldsymbol{s}}(z^{-1})$ has interesting consequences. Besides providing computational advantages in the numerical solution of matrix equations, it allows one to give an explicit expression for the solution of the Poisson problem for QBDs [2]. Another interesting issue related to the shift technique concerns conditioning. In fact, while null recurrent problems are ill-conditioned, the shifted counterparts are not. A convenient computational strategy to solve a null recurrent problem consists in transforming it into a new one, say by means of the double shift; solve the latter by using a quadratic convergent algorithm like cyclic reduction or logarithmic reduction [3]; then recover the solution of the original problem from the one of the shifted problem. For this conversion, the expressions relating the solutions of the shifted equations to those of the original equations are fundamental, they are provided in this paper.

Extensions of the shift technique to matrix polynomials of any degree, to matrix power series and Laurent matrix series, where the coefficients do not necessarily satisfy sign properties, are given in [4], where some experiments are presented concerning algorithmic improvements.

The paper is organized as follows. In Section 2 we recall some properties of the canonical factorization of matrix polynomials, and their interplay with the solutions of the associated quadratic matrix equations, with specific attention to those equations encountered in QBD processes. In Section 3 we present the shift techniques in functional form, with attention to the properties of the roots of the original and modified matrix polynomial. In Section 4 we state the main results on the existence and properties of the canonical factorizations. In particular we provide explicit relations between the solutions of the original matrix equations and the solutions of the shifted equations. In the Appendix, the reader can find the proof of a technical property used to prove the main results.

2. Preliminaries

In this section we recall some properties of matrix polynomials and of QBDs, that will be used later in the paper. For a general treatment on these topics we refer to the books [3,9,14,18,20].

2.1. Matrix polynomials

Consider the Laurent matrix polynomial $\varphi(z) = \sum_{i=-1}^{1} z^i B_i$, where B_i, $i = -1, 0, 1$, are $n \times n$ complex matrices. A canonical factorization of $\varphi(z)$ is a decomposition of the kind $\varphi(z) = E(z)F(z^{-1})$, where $E(z) = E_0 + zE_1$ and $F(z) = F_0 + zF_{-1}$ are invertible for $|z| \le 1$. A canonical factorization is *weak* if $E(z)$ and $F(z)$ are invertible for $|z| < 1$ but possibly singular for some values of z such that $|z| = 1$. The canonical factorization is unique in the form $\varphi(z) = (I - z\widetilde{E}_1)K(I - z^{-1}\widetilde{F}_{-1})$ for suitable matrices $\widetilde{E}_1$, $\widetilde{F}_{-1}$ and K, see for instance [8].

Given an $n \times n$ quadratic matrix polynomial $B(z) = B_{-1} + zB_0 + z^2 B_1$, we call *roots of* $B(z)$ the roots $\xi_1, \ldots, \xi_{2n}$ of the polynomial $\det B(z)$ where we assume that there are k roots at infinity if the degree of $\det B(z)$ is $2n - k$. In the sequel we also assume that the roots are ordered so that $|\xi_1| \le \cdots \le |\xi_{2n}|$.

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