Applied Numerical Mathematics ••• (••••) •••–•••

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Shanks function transformations in a vector space

Claude Brezinski [a], Michela Redivo-Zaglia [b,*]

[a] Université de Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
[b] Università degli Studi di Padova, Dipartimento di Matematica, Via Trieste 63, 35121–Padova, Italy

ARTICLE INFO

Article history:
Available online xxxx

Friendly dedicated to Prof. Francesco Costabile on the occasion of his 70th birthday.

Keywords:
Function transformation
Shanks transformation
Confluent ε-algorithm
Lotka–Volterra equation

ABSTRACT

In this paper, we show how to construct various extensions of Shanks transformation for functions in a vector space. They are aimed at transforming a function tending slowly to its limit when the argument tends to infinity into another function with better convergence properties. Their expressions as ratio of determinants and recursive algorithms for their implementation are given. A simplified form of one of them is derived. It allows us to obtain a convergence result for an important class of functions. An application to integrable systems is discussed.

1. Introduction

In numerical analysis and in applied mathematics, many methods make use of sequences. If a sequence is slowly converging, it can be useful to accelerate it. This can be done, in some cases, by modifying the process which constructs the sequence. However, if this process is a black box and if one has no access to it, another possibility is to transform the sequence, by a so-called *sequence transformation*, into another sequence converging faster to the same limit under some assumptions. For sequences of numbers, one of the most well-known such transformations is due to Shanks [12]. It can be recursively implemented by the scalar ε-algorithm of Wynn [13]. This algorithm was extended by Wynn to sequences of vectors [15]. But, since the algebraic theory underlying this *vector ε-algorithm* could not be easily derived from its rule, two other generalizations were proposed in [3]. The first one can be recursively implemented by two different *topological ε-algorithms* (TEA1 and TEA2), while the second one requires the use of the $S\beta$-algorithm [10] (see also [11]). Recently, the rules of the TEA1 and TEA2 were greatly simplified, thus leading to the *simplified topological ε-algorithms* (STEA1 and STEA2) [7]. The corresponding software and applications can be found in [8].

Similarly, when a real (or complex) function of a real (or complex) variable is tending slowly to a limit with its argument, it can be transformed into a new function by a *function transformation*. Function transformations are usually obtained by replacing the divided differences appearing in a scalar sequence transformation by derivatives after letting a parameter tend to zero. This is why the adjective *confluent* was added to the name of the algorithms for implementing them. A Shanks (real or complex) function transformation was presented by Wynn together with the corresponding *confluent ε-algorithm* for its implementation [14]. Let us mention that generalizations of this transformation and of this algorithm to the confluent vector case were never proposed. Later, a topological Shanks function transformation for functions in a vector space, together with its confluent topological ε-algorithm, were proposed in [4].

* Corresponding author.
E-mail addresses: Claude.Brezinski@univ-lille1.fr (C. Brezinski), Michela.RedivoZaglia@unipd.it (M. Redivo-Zaglia).

http://dx.doi.org/10.1016/j.apnum.2016.06.013
0168-9274/

Please cite this article in press as: C. Brezinski, M. Redivo-Zaglia, Shanks function transformations in a vector space, Appl. Numer. Math. (2016), http://dx.doi.org/10.1016/j.apnum.2016.06.013

The aim of this paper is to consider several topological Shanks function transformations in a vector space and to discuss the implementation of some of them by confluent recursive algorithms. A new and cheaper confluent topological ε-algorithm is presented. This new algorithm allows us to obtain a convergence theorem for an important class of functions. An application to integrable systems is discussed.

2. Topological Shanks function transformations

Let $t \longmapsto \mathbf{f}(t) \in E$ be a function depending on a parameter $t \in D \subseteq \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), where E is a vector space of sufficiently differentiable functions on $\mathbb{K}$. In many practical situations, E is $\mathbb{R}^p$ or $\mathbb{R}^{p\times q}$. We want to construct, for a fixed value of k, a *function transformation* $\mathbf{f} \in E \longmapsto \mathbf{e}_k \in E$ such that, for all t, $\mathbf{e}_k(t) = \mathbf{S} \in E$ if the function $\mathbf{f}$ satisfies the linear differential equation of order k, called the *kernel* of the transformation,

$$a_0(\mathbf{f}(t) - \mathbf{S}) + a_1\mathbf{f}'(t) + \cdots + a_k\mathbf{f}^{(k)}(t) = \mathbf{0}, \quad \forall t \tag{1}$$

where $\mathbf{S} \in E$ is independent of t, and $a_0, \ldots, a_k \in \mathbb{K}$ are independent of k and t. Obviously, a_0 has to be different from zero and it does not restrict the generality to impose the *normalization condition* $a_0 = 1$. In the case of a complex or real function, such a transformation was proposed by Wynn [14] together with the corresponding recursive algorithm, the confluent ε-algorithm, for its implementation. We will now generalize this transformation and this algorithm to a vector space E.

In Section 2.1, we consider a kernel of the form (1) leading to two transformations, while, in Section 2.3, we will discuss another form of the kernel giving rise to two other transformations.

2.1. First topological Shanks function transformations

Obviously if the function $\mathbf{f} \in E$ satisfies a relation of the form (1) where the coefficients a_i are known, such a function transformation is simply defined by

$$\mathbf{e}_k(\mathbf{f}(t)) = \mathbf{f}(t) + a_1\mathbf{f}'(t) + \cdots + a_k\mathbf{f}^{(k)}(t), \quad \forall t, \tag{2}$$

since $a_0 = 1$, and it holds, $\forall t$, $\mathbf{e}_k(\mathbf{f}(t)) = \mathbf{S}$.

Let us now see how to compute the coefficients a_i when they are unknown. Differentiating (1), we have, for all t

$$a_0\mathbf{f}'(t) + \cdots + a_k\mathbf{f}^{(k+1)}(t) = \mathbf{0}.$$

As in the case of sequences, we have to transform this relation in E into a relation in $\mathbb{K}$. Taking the duality product with $\mathbf{y} \in E^*$ (the algebraic dual space of E, that is the space of linear functionals on E), we obtain

$$a_0\langle \mathbf{y}, \mathbf{f}'(t)\rangle + \cdots + a_k\langle \mathbf{y}, \mathbf{f}^{(k+1)}(t)\rangle = 0.$$

Then, several possibilities occur. Considering this relation, differentiating $k-1$ times the functions appearing in it, and taking the normalization condition into account leads to a system of $k+1$ equations in the unknowns $a_0, \ldots, a_k$ (case 1). Instead of differentiating several times the preceding relation, another way is to use k different linear functionals $\mathbf{y}_1, \ldots, \mathbf{y}_k \in E^*$ (case 2). Obviously, a mixture of these two strategies could also be considered but it leads to a more complicated transformation and it will not be considered further.

In both cases, adding the normalization condition, a system of $k+1$ equations in $k+1$ unknowns is obtained, and, by construction, the transformation (2) gives $\forall t$, $\mathbf{e}_k(\mathbf{f}(t)) = \mathbf{S}$.

Let us now apply the same procedure to a function which does not belong to the kernel of the transformation. Any of the preceding strategies for constructing the algebraic system giving the coefficients a_i can still be used. However, since its solution now depends on k it will be designated by $a_i^{(k)}$ for $i = 0, \ldots, k$. Then, in both cases, we define the *first topological Shanks function transformation* by

$$\mathbf{e}_k(\mathbf{f}(t)) = a_0^{(k)}\mathbf{f}(t) + a_1^{(k)}\mathbf{f}'(t) + \cdots + a_k^{(k)}\mathbf{f}^{(k)}(t), \tag{3}$$

with $a_0^{(k)} = 1$, and, by construction, it holds

Theorem 1. *If, for all t,*

$$\mathbf{f}(t) = \mathbf{S} - a_1\mathbf{f}'(t) - \cdots - a_k\mathbf{f}^{k}(t),$$

then, for all t,

$$\mathbf{e}_k(\mathbf{f}(t)) = \mathbf{S}.$$

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