Applied Numerical Mathematics 116 (2017) 129–140

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A class of Birkhoff–Lagrange-collocation methods for high order boundary value problems

Francesco A. Costabile, Anna Napoli *

Department of Mathematics and Informatics, University of Calabria, 87036 Rende (CS), Italy

ARTICLE INFO

Article history:
Available online 16 December 2016

Keywords:
Boundary value problem
Collocation methods
Interpolation

ABSTRACT

A general procedure to determine collocation methods for high order boundary value problems is presented. These methods provide globally continuous differentiable solution in the form of polynomial functions and also numerical solution on a set of discrete points. Some special cases are considered. Numerical experiments are presented which support theoretical results and provide favorable comparisons with other existing methods.

1. Introduction

Let us consider the general r-th order ordinary differential equation (ODE)

$$y^{(r)}(x) = f\left(x, \mathbf{y}(x)\right), \qquad x \in I = [a, b], \quad r > 1 \tag{1}$$

where $\mathbf{y}(x) = \left(y(x), y'(x), \ldots, y^{(q)}(x)\right), 0 \le q < r$. We associate with equation (1) the r boundary conditions

$$B[y](x) = g(x), \qquad g \in \mathbb{R}^r, \; x \in \partial I \tag{2}$$

where B is a linear operator.

We suppose that $f : [a, b] \times \mathbb{R}^{q+1} \to \mathbb{R}$ is continuous at least in the interior of the domain of interest and it satisfies a uniform Lipschitz condition in $\mathbf{y}$, which means that there exists a nonnegative constant L such that, whenever $(x, y_0, y_1, \ldots, y_q)$ and $(x, \overline{y}_0, \overline{y}_1, \ldots, \overline{y}_q)$ are in the domain of f, the inequality

$$\left|f\left(x, y_0, y_1, \ldots, y_q\right) - f\left(x, \overline{y}_0, \overline{y}_1, \ldots, \overline{y}_q\right)\right| \le L \sum_{k=0}^{q} \left|y_k - \overline{y}_k\right| \tag{3}$$

holds. Moreover, we assume that the conditions for the existence and uniqueness of solution of problem (1)–(2) in a certain appropriate domain of $[a, b] \times \mathbb{R}^{q+1}$ are satisfied [1] and that the solution $y(x)$ is differentiable with continuity up to what is necessary.

Problems of these kinds model a wide spectrum of nonlinear phenomena. In fact, ordinary (and also partial) differential equations arise in the study of astrophysics, hydrodynamic and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, engineering and applied physics (see [7,21,23,30,31] and references therein).

* Corresponding author.
E-mail addresses: costabile@unical.it (F.A. Costabile), anna.napoli@unical.it (A. Napoli).

http://dx.doi.org/10.1016/j.apnum.2016.12.003
0168-9274/

Finite difference and finite element methods (see for instance [5,8,22]) have long histories as particularly flexible and powerful methods for the numerical solution of differential equations. In the last decades spectral methods (see [6,24,28] and references therein) have emerged as a valid alternative to those methods, being very much successful and extremely accurate for the numerical solution of ordinary or partial differential equations. They make use of global representations, usually by high-order polynomials or Fourier series. Spectral methods have been intensively and increasingly studied, especially, since the development of fast transform methods, with applications in problems where high accuracy is required. Theoretical studies and numerical experience have confirmed that for problems with smooth solutions they converge faster than finite difference and finite element methods and with a degree of accuracy that local methods cannot match. Summary of some applications is given in [24] (see also references therein). The basic idea of spectral methods is to use a set of basis functions ϕ_i, $i = 0, \ldots, N$, also called trial or expansion approximating functions (very smooth and global such as polynomials), to represent the solution to a problem as a truncated series $y_N(x) = \sum_{i=0}^{N} a_i \phi_i(x)$, $x \in (a, b)$ where a_i are the unknown coefficients. A spectral method is characterized by a specific way to determine these coefficients. For instance, in *collocation* methods (also called pseudo-spectral) the numerical approximation $y_N(x)$ of the solution $y(x)$ to a problem is required to satisfy the given differential equation in a discrete set of points, called collocation nodes, of the interval (a, b). The choice of appropriate spectral representation depends on the kind of initial or boundary conditions involved in the problem.

More recently, in [32] the authors, taking up an idea in [10–18,20], proved that the method, based on a Birkhoff type interpolation [15], is well conditioned.

In this paper we will give a unified theory of collocation methods based on Birkhoff–Lagrange interpolation, already proposed by the authors and others (see [10,12–14,16–18,20,32]).

The outline of the paper is the following: in Section 2 we derive the new class of methods and we give an a priori estimation of the global error. In Section 3, in order to implement the presented methods, we propose an algorithm to compute the numerical solution of (1)–(2) in a set of nodes. Then, in Section 4 we consider some special cases of the considered problem which include methods already present in the literature. The analysis is accompanied by numerical examples (Section 5), which support theoretical results. Comparison with other existing methods is also given.

2. The Birkhoff–Lagrange collocation methods

Preliminary, we prove that problem (1)–(2) is equivalent to a Fredholm integral equation. We observe that (2) is a linear interpolation problem. Suppose that it has a unique solution, that is, there exists a unique polynomial of degree $r-1$, $P_{r-1}[y](x) \in \mathcal{P}_{r-1}$ which satisfies

$$B\left[P_{r-1}\right](x) = g, \quad x \in \partial I. \tag{4}$$

Moreover, if $y \in C^r(I)$, then

$$y(x) = P_{r-1}[y](x) + \int_a^b K_r^x(x,t)\, y^{(r)}(t))\, dt \tag{5}$$

where

$$K_r^x(x,t) = \frac{1}{(r-1)!}\left[(x-t)_+^{r-1} - P_{r-1}\left[(x-t)_+^{r-1}\right](x)\right] \tag{6}$$

is the Peano kernel of the remainder in the considered interpolation problem, and $(x-t)_+ = x-t$ if $x \geq t$, $(x-t)_+ = 0$ if $x < t$.

Remark 1. From (5), (2) and (4) it follows that $B\left[K_r^x\right](x) = 0, x \in \partial I$.

Theorem 1. *In the above hypothesis and notations the boundary value problem (1)–(2) is equivalent to the Fredholm integral equation*

$$y(x) = P_{r-1}[y](x) + \int_a^b K_r^x(x,t)\, f\,(t, \mathbf{y}(t))\, dt \tag{7}$$

where $P_{r-1}[y](x)$ and $K_r^x(x,t)$ are, respectively, the unique polynomial solution and the corresponding Peano kernel in the interpolatory problem (2).

Proof. Let $y(x)$ be the solution of (1)–(2). Hence the linear interpolatory problem (2) has a unique solution, that is, there exists a unique polynomial $P_{r-1}[y](x)$ of degree $r-1$ and the related Peano kernel $K_r^x(x,t)$ such that (5) holds. The (7) follows tacking into account (1).

Now, suppose that $y(x)$ is solution of (7). Under the hypothesis on $P_{r-1}[y]$ and $K_r^x(x,t)$, relations (2) and (5) hold. Therefore, from (5), (7) and the uniqueness of the Peano Kernel, the (1) follows.