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C^1 quintic splines on domains enclosed by piecewise conics and numerical solution of fully nonlinear elliptic equations

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ABSTRACT

We introduce bivariate C^1 piecewise quintic finite element spaces for curved domains enclosed by piecewise conics satisfying homogeneous boundary conditions, construct local bases for them using Bernstein–Bézier techniques, and demonstrate the effectiveness of these finite elements for the numerical solution of the Monge–Ampère equation over curved domains by Böhmer's method.

1. Introduction

Piecewise polynomials on curved domains bounded by piecewise algebraic curves and surfaces is a promising but little studied tool for data fitting and solution of partial differential equations. Since implicit algebraic surfaces are a well-established modeling technique in CAD [6], we are interested in developing isogeometric schemes [21] for domains with such boundaries, where the geometric models of the boundary are used exactly in the form they exist in a CAD system rather than undergoing a remeshing to fit into the traditional isoparametric finite element approach.

In this paper we continue the work started in [14], where C^0 splines vanishing on a piecewise conic boundary have been introduced. In contrast to both the isoparametric curved finite elements and the isogeometric analysis of [21], our approach does not require parametric patching on curved subtriangles, and hence does not depend on the invertibility of the Jacobian matrices of the nonlinear geometry mappings. Therefore our finite elements remain piecewise polynomial everywhere in the physical domain.

This approach allows to incorporate conditions of higher smoothness in Bernstein–Bézier form standard for the theory and practice of smooth piecewise polynomials on polyhedral domains [22]. It turns out however that imposing boundary conditions make the otherwise well understood spaces of e.g. bivariate C^1 macro-elements on triangulations significantly more complex. Even in the simplest case of a polygonal domain, the dimension of the space of splines vanishing on the boundary is dependent on its geometry, with consequences for the construction of stable bases (or stable minimal determining sets) [15,16].

In this paper we suggest a local basis defined through a minimal determining set for the space of C^1 piecewise quintic polynomials vanishing on a piecewise conic boundary and apply the resulting finite element space to the numerical solution of the fully nonlinear Monge–Ampère equation on domains with such boundary. The latter is done within the framework of Böhmer's method [7]. The results are based in part on the thesis of the second named author [26].

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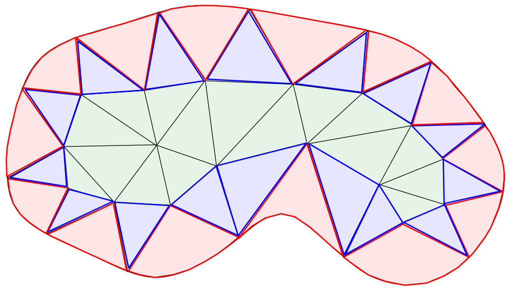

Fig. 1. A triangulation of a curved domain with ordinary triangles (green), pie-shaped triangles (pink) and buffer triangles (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Böhmer's method requires stable local bases of C^1 piecewise polynomials vanishing on the boundary. In an earlier paper [16] we developed such bases on polygonal domains and demonstrated their effectiveness in solving fully nonlinear elliptic equations. Further details on Böhmer's method and references to the literature on this subject are given in Section 4.1. Note that to the best of our knowledge no method for fully nonlinear equations has been tested before on curved domains.

It is important to mention that the isoparametric approach to C^0 curved elements is problematic when finite element spaces of C^1 or higher smoothness are sought, see the remarks in [10, Section 4.7]. A successful C^1 quintic construction of this type developed in [5] seems difficult to extend to higher smoothness or higher polynomial degree. Similar difficulties to achieve C^1 or higher smoothness have recently been reported for the tensor-product based isogeometric analysis as soon as more than one patch is needed to model the geometry, see e.g. [11]. This is expected because isogeometric analysis also employs "isoparametric mappings," with tensor-product spline spaces replacing polynomials.

Remarkably, the standard Bernstein–Bézier techniques for dealing with piecewise polynomials on triangulations [22,27] as well as recent optimal assembly algorithms [1–3] for high order elements are carried over to the spaces used here without significant loss of efficiency, see [14].

The paper is organized as follows. The spaces of C^1 piecewise polynomials on domains with piecewise conic boundary are introduced in Section 2, whereas Section 3 presents our construction of a local basis for the main space of interest $S^{1,2}_{5,0}(\triangle)$. Section 4 briefly summarizes Böhmer's method for fully nonlinear elliptic equations and presents a number of numerical experiments for the Monge–Ampère equation on smooth domains, including a circular domain, an elliptic domain, and piecewise conic domains with C^1 and C^2 boundaries.

2. C^1 piecewise polynomials on piecewise conic domains

We first recall from [14] the assumptions on a domain Ω and its triangulation $\triangle$ with curved pie-shaped triangles at the boundary.

Let $\Omega \subset \mathbb{R}^2$ be a bounded curvilinear polygonal domain with $\Gamma = \partial\Omega = \bigcup_{j=1}^{n} \overline{\Gamma}_j$, where each Γ_j is an open arc of an algebraic curve of at most second order (i.e., either a straight line or a conic). For simplicity we assume that Ω is simply connected. Let $Z = \{z_1, \ldots, z_n\}$ be the set of the endpoints of all arcs numbered counter-clockwise such that z_j, z_{j+1} are the endpoints of Γ_j, $j = 1, \ldots, n$, with $z_{j+n} = z_j$. Furthermore, for each j we denote by ω_j the internal angle between the tangents τ_j^+ and τ_j^- to Γ_j and Γ_{j-1}, respectively, at z_j. We assume that $\omega_j > 0$ for all j.

Let $\triangle$ be a *triangulation* of Ω, i.e., a subdivision of Ω into triangles, where each triangle $T \in \triangle$ has at most one edge replaced with a curved segment of the boundary $\partial\Omega$, and the intersection of any pair of the triangles is either a common vertex or a common (straight) edge if it is non-empty. The triangles with a curved edge are said to be *pie-shaped*. Any triangle $T \in \triangle$ that shares at least one edge with a pie-shaped triangle is called a *buffer* triangle, and the remaining triangles are *ordinary*. We denote by $\triangle_0$, $\triangle_B$ and $\triangle_P$ the sets of all ordinary, buffer and pie-shaped triangles of $\triangle$, respectively, such that $\triangle = \triangle_0 \cup \triangle_B \cup \triangle_P$ is a disjoint union, see Fig. 1. Let V, E, V_I, E_I, V_B, E_B denote the set of all vertices, all edges, interior vertices, interior edges, boundary vertices and boundary edges, respectively.

For each $j = 1, \ldots, n$, let $q_j \in \mathbb{P}_2$ be a polynomial such that $\Gamma_j \subset \{x \in \mathbb{R}^2 : q_j(x) = 0\}$, where $\mathbb{P}_d$ denotes the space of all bivariate polynomials of total degree at most d. By changing the sign of q_j if needed, we ensure that $\partial_{\nu_x} q_j(x) < 0$ for all x in the interior of Γ_j, where ν_x denotes the unit outer normal to the boundary at x, and $\partial_a := a \cdot \nabla$ is the directional derivative with respect to a vector a. Hence, $q_j(x)$ is positive for points in Ω near the boundary segment Γ_j. We assume that $q_j \in \mathbb{P}_1$ if Γ_j is a straight interval. Clearly, q_j is an irreducible quadratic polynomial if Γ_j is a genuine conic arc and in all cases

$$\nabla q_j(x) \neq 0 \quad \text{if} \quad x \in \Gamma_j. \tag{1}$$

Following [14] we assume that $\triangle$ satisfies the following conditions:

(a) $Z = \{z_1, \ldots, z_n\} \subset V_B$.
(b) No interior edge has both endpoints on the boundary.
(c) No pair of pie-shaped triangles shares an edge.