



Scattering data computation for the Zakharov–Shabat system with non-smooth potentials



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ABSTRACT

In this paper we present a variant of the method for the scattering data computation for the Zakharov–Shabat system, recently proposed by the authors. The algorithm that characterizes this variant allows us to compute the scattering data also in the presence of jump discontinuities of the initial potential.

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1. Introduction

Over the last 50 years, due to their relevance in applications, much research has been conducted in the field of nonlinear partial differential equations (NPDEs) of integrable type [1,2]. Among them, a special role is played by the nonlinear Schrödinger (NLS) equation which governs the signal transmission in fiber optics [5,7], as well as in surface waves on deep water [1,2].

The initial value problem (IVP) for the NLS equation can be formulated as follows:

$$\begin{cases} \mathbf{i}u_t + u_{xx} \pm 2|u|^2u = 0, & x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \quad (1)$$

where \mathbf{i} denotes the imaginary unit, $u = u(x, t)$ is the unknown potential, the subscripts x and t designate partial derivatives with respect to position and time, $u_0 \in L^1(\mathbb{R})$ is the initial potential and the \pm sign depends on symmetry properties of u . The plus sign regards the focusing case and the minus sign the defocusing case. As Zakharov and Shabat proved [11], the IVP for the NLS equation can be associated to the first order system of ordinary differential equations

$$\mathbf{i}j \frac{\partial \Psi}{\partial x}(\lambda, x) - \mathbf{V}(x)\Psi(\lambda, x) = \lambda \Psi(\lambda, x), \quad x \in \mathbb{R} \quad (2)$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{V} = \mathbf{i} \begin{pmatrix} 0 & u_0 \\ v_0 & 0 \end{pmatrix} \quad (3)$$

with $v_0 = u_0^*$ in the focusing case and $v_0 = -u_0^*$ in the defocusing case. Here and in the sequel the asterisk denotes the complex conjugate.

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With the help of this system, known as the ZS system, the solution of (1) can theoretically be obtained from the initial potential u_0 by means of the so-called Inverse Scattering Transform (IST) technique.

An effective method to compute all the scattering data for the ZS system has been recently developed under the hypothesis that $u_0 \in C^0(\mathbb{R})$ [4]. In this paper we propose a variant of this method that allows us to apply the method even in the case $u_0 \notin C^0(\mathbb{R})$.

The paper is organized as follows. In Section 2 we recall the definition of the scattering data and the properties needed to the illustration of our method. In Section 3 we recall the five steps for computing the scattering data [4] under the assumption that $u_0 \in C^0(\mathbb{R})$. The technique that allows us to extend the method to the case $u_0 \notin C^0(\mathbb{R})$ is illustrated in Section 4. In Section 5 we consider an initial potential with a discontinuity jump to illustrate the effectiveness of our method. Finally Section 6 is devoted to the conclusions.

2. Scattering data: definitions and properties

We start by recalling the crucial role played in the computation of the scattering data by the Jost solutions [2], that is by the solutions of the ZS system (2) which satisfy the asymptotic conditions

$$(\bar{\Psi}(\lambda, x), \Psi(\lambda, x)) = e^{-i\lambda Jx}(I + o(1)), \quad x \rightarrow +\infty \tag{4}$$

$$(\Phi(\lambda, x), \bar{\Phi}(\lambda, x)) = e^{-i\lambda Jx}(I + o(1)), \quad x \rightarrow -\infty \tag{5}$$

where $\lambda \in \mathbb{R}$, I denotes the identity matrix and J is defined in (3).

Since these solutions satisfy the same linear first order system, there exist transition matrices

$$\mathbf{A}_\ell(\lambda) = \begin{pmatrix} a_{\ell 1}(\lambda) & a_{\ell 2}(\lambda) \\ a_{\ell 3}(\lambda) & a_{\ell 4}(\lambda) \end{pmatrix}, \quad \mathbf{A}_r(\lambda) = \mathbf{A}_\ell(\lambda)^{-1} = \begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{pmatrix} \tag{6}$$

such that

$$(\bar{\Psi}(\lambda, x), \Psi(\lambda, x)) = (\Phi(\lambda, x), \bar{\Phi}(\lambda, x))\mathbf{A}_\ell(\lambda)$$

$$(\Phi(\lambda, x), \bar{\Phi}(\lambda, x)) = (\bar{\Psi}(\lambda, x), \Psi(\lambda, x))\mathbf{A}_r(\lambda).$$

Denoting by \mathbb{C}^+ and \mathbb{C}^- the upper and lower half plane and by $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$ their closures, respectively, the following analytic properties hold true. The Jost functions $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ are continuous in $\lambda \in \bar{\mathbb{C}}^+$, are analytic in $\lambda \in \mathbb{C}^+$, and behave as $e^{-i\lambda x J}$ as $\lambda \rightarrow \infty$ in $\bar{\mathbb{C}}^+$, whereas $\bar{\Psi}(\lambda, x)$ and $\bar{\Phi}(\lambda, x)$ are continuous in $\lambda \in \bar{\mathbb{C}}^-$, are analytic in $\lambda \in \mathbb{C}^-$, and behave as $e^{-i\lambda x J}$ as $\lambda \rightarrow \infty$ in $\bar{\mathbb{C}}^-$. We can then rewrite (4) and (5) as the Riemann–Hilbert problem

$$(\bar{\Psi}(\lambda, x), \bar{\Phi}(\lambda, x)) = (\Phi(\lambda, x), \Psi(\lambda, x))\mathbf{J}\mathbf{S}(\lambda)\mathbf{J}$$

where

$$\mathbf{S}(\lambda) = \begin{pmatrix} T(\lambda) & L(\lambda) \\ R(\lambda) & T(\lambda) \end{pmatrix}. \tag{7}$$

In (7) $T(\lambda)$ represents the transmission coefficient, $R(\lambda)$ denotes the reflection coefficients from the right and $L(\lambda)$ stands for the reflection coefficients from the left.

If $T(\lambda)$ has no poles in the complex upper half plane \mathbb{C}^+ (as occurs in the defocusing case), the transmission coefficient and the reflection coefficients are the only scattering data to identify. Otherwise, denoting by $\lambda_1, \dots, \lambda_n$ the so-called bound states, that is the finitely many poles of $T(\lambda)$ in \mathbb{C}^+ , and by m_1, \dots, m_n the corresponding multiplicities, we have to identify the parameters $\{n, m_j, \lambda_j\}$ as well as the coefficients $\{(\Gamma_\ell)_{js}, (\Gamma_r)_{js}\}$ of the spectral sums from the left and from the right

$$S_\ell(\alpha) = \sum_{j=1}^n e^{i\lambda_j \alpha} \sum_{s=0}^{m_j-1} (\Gamma_\ell)_{js} \frac{\alpha^s}{s!}, \quad \alpha \geq 0,$$

$$S_r(\alpha) = \sum_{j=1}^n e^{i\lambda_j^* \alpha} \sum_{s=0}^{m_j-1} (\Gamma_r)_{js} \frac{\alpha^s}{s!}, \quad \alpha \leq 0,$$

where $0! = 1$.

The method developed in [4] allows us to compute all the scattering data, i.e. the spectral matrix \mathbf{S} introduced in (7) as well as the discrete scattering data $\{\lambda_j, (\Gamma_\ell)_{js}, (\Gamma_r)_{js}\}$ whenever $S_\ell(\alpha)$ and $S_r(\alpha)$ are known in $N \geq M = m_1 + \dots + m_n$ points.

To this end we need to compute the Marchenko kernels from the left $\Omega_\ell(\alpha)$ and from the right $\Omega_r(\alpha)$. These two kernels are, in fact, connected to the above spectral coefficients and spectral sums as follows:

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