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A numerical method for the solution of integral equations of Mellin type

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A R T I C L E I N F O

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Dedicated to Professor Francesco A. Costabile on the occasion of his 70th birthday

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A B S T R A C T

We are interested in the numerical solution of second kind integral equations of Mellin convolution type. We describe a modified Nyström method based on the Gauss–Lobatto or Gauss–Radau quadrature rule. Under certain assumptions on the Mellin kernel, we prove the stability and the convergence of the proposed procedure and also derive error estimates. Finally, some test problems are solved and the numerical results showing the effectiveness of our method are presented.

1. Introduction

We are interested in the numerical solution of second kind integral equations of the following type

$$f(y) + \int_0^1 [K(x, y) + H(x, y)] f(x)dx = g(y), \quad 0 < y \le 1, \tag{1}$$

where $K(x, y)$ is a Mellin kernel, continuous for all $x + y > 0$ and such that

$$K(x, y) = \pm \frac{1}{x} k\left(\frac{y}{x}\right), \tag{2}$$

for some given function $k : [0, \infty) \to [0, \infty)$ satisfying the following assumption

$$\int_0^\infty \frac{k(t)}{t} dt < \infty, \tag{3}$$

$H(x, y)$ and $g(y)$ are given continuous functions, and $f(x)$ is the unknown.

The mathematical formulation of many problems in physics and engineering leads to solve integral equations of the form (1). For instance, they occur when boundary integral methods are applied in order to solve potential problems on planar domains with corners or crack problems in linear elasticity (see [1,3,16] and the references therein).

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Representing the solution of the interior Dirichlet problem on a domain with piecewise smooth boundary in the form of a double layer potential leads to a system of integral equations involving Mellin convolution operators with kernels $K(x,y)$ of the form

$$K(x,y) = \frac{1}{\pi}\frac{y\sin(\chi\pi)}{x^2+2xy\cos(\chi\pi)+y^2} \tag{4}$$

with $(1-\chi)\pi$, $\chi \in (-1,1)$, the interior angle at the corner point (see [2,11]).

The kernel

$$K(x,y) = \frac{4}{\pi}\frac{xy^2}{(x^2+y^2)^2} \tag{5}$$

arises from a problem of determining the distribution of stress in a thin elastic plate in the vicinity of a cruciform crack (see [16,14,17]).

In general the solution $f(y)$ of (1) (or perhaps its higher derivatives) will have a singularity at $y=0$. However, when proposing a numerical method for it, the main difficulty one encounters is the proof of its stability, since, while the integral operators

$$(\mathcal{K}f)(y) = \int_0^1 K(x,y)f(x)dx \tag{6}$$

and

$$(\mathcal{H}f)(y) = \int_0^1 H(x,y)f(x)dx \tag{7}$$

are both bounded maps on the space $C[0,1]$, only the operator $\mathcal{H}$ is compact. The Mellin convolution operator $\mathcal{K}$ is not compact being its kernel $K(x,y)$ not smooth or weakly singular on $[0,1]\times[0,1]$, but containing a fixed first order singularity at $x=y=0$.

When we are interested in the numerical solution of equation (1) by means of Nyström or discrete collocation methods, the evaluation of the integral transform of Mellin type $\mathcal{K}f$ at some chosen collocation points represents a crucial step. Hence efficient quadrature formulas are necessary in order to approximate the integrals $(\mathcal{K}f)(y)$, $y \in (0,1]$.

In this paper, at first, we propose an algorithm for the evaluation of such integrals, since the fixed singularity of the Mellin kernel $K(x,y)$ at the origin could make inefficient the use of the classical Gaussian rules when y is very close to the endpoint 0. Then, following [14,15], we propose a "modified" Nyström type method for the numerical solution of (1).

In [14] the Authors propose to modify the Gauss–Legendre or a suitable product quadrature rule in order to construct a stable Nyström type approximant for the solution of a particular convolution equation, whose Mellin kernel is given by (5) and whose right hand side and solution vanish at the origin.

In [15] a Nyström method for the solution of more general Mellin convolution integral equations, i.e. equations of type (1), with $K(x,y) = \pm\frac{1}{x}k\left(\frac{y}{x}\right)$ satisfying (3) and $H(x,y)\equiv 0$, is proposed. It is based on a product quadrature formula constructed by replacing the function $f(x)$ by its Lagrange polynomial associated with the nodes of the Gauss–Radau quadrature rule. The rule is modified near the origin in order to prove the stability of the Nyström interpolant. The case of solutions non-vanishing at the origin is also considered here. The proposed procedure can be applied once one has determined the value $f(0)$, and this can be easily done if the integral in (3) is known.

Our method is based on a slight modification of the classical Gauss–Radau or Gauss–Lobatto rules. These modified quadrature formulas give rise to stable and convergent procedures for the numerical solution of Mellin integral equations of type (1), which can be employed in the more general case where the solution $f(x)$ does not necessarily vanish at the origin and $H(x,y)\not\equiv 0$.

Let us remark that when $f(0)\neq 0$ it is not possible to use the modified Gauss–Legendre rule considered in [14] as well as when $H(x,y)\not\equiv 0$ the approach described in [15] cannot be followed. The stability and convergence analysis performed in [15] does not apply anymore. In such a case only the Radau and Lobatto ones can be used. In this way, since one of the quadrature nodes coincides with the interval endpoint 0, the value $f(0)$ turns out to be one of the final linear system unknowns. We also prove that this linear system is well conditioned.

Very recently a different modified Nyström type method has been proposed in [4] for the numerical treatment of integral equations having the form (1), but with the Mellin kernel $K(x,y)$ in (2) satisfying the condition

$$\int_0^\infty \frac{k(t)}{t^{\frac{1}{2}}}dt < \infty \tag{8}$$

instead of (3). In this case the Mellin operator $\mathcal{K}$ defined in (6) is a continuous map from $L^2[0,1]$ into itself but it could be not bounded with respect to the uniform norm. As a consequence, the solution f of (1) could be singular at the origin.

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