Applied Numerical Mathematics 116 (2017) 230–237

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Stability and boundedness of numerical approximations to Volterra integral equations

Eleonora Messina [a,*], Antonia Vecchio [b]

[a] Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli "Federico II", Via Cintia, I-80126 Napoli, Italy
[b] C.N.R. National Research Council of Italy, Institute for Computational Application "Mauro Picone", Via P. Castellino, 111, 80131 Napoli, Italy

ARTICLE INFO

Article history:
Available online 26 January 2017

Keywords:
Volterra Integral Equations
Numerical stability
Direct quadrature methods

ABSTRACT

Volterra Integral Equations (VIEs) arise in many problems of real life, as, for example, feedback control theory, population dynamics and fluid dynamics. A reliable numerical simulation of these phenomena requires a careful analysis of the long time behavior of the numerical solution. Here we develop a numerical stability theory for Direct Quadrature (DQ) methods which applies to a quite general and representative class of problems. We obtain stability results under some conditions on the stepsize and, in particular cases, unconditional stability for DQ methods of whatever order.

1. Introduction

In the mathematical representation of real life history-dependent problems (such as mechanical systems, electric circuit, epidemiology, population growth,...), systems of Volterra integral equations are widely used (see e.g., [5,9–11]) and have been subject of extensive investigations over the years (see [5,6,10] and the bibliography therein). Here, we are concerned in the numerical simulations of linear VIEs of the type

$$y(t) = f(t) + \int_0^t k(t,s)\,y(s)ds,\ t \geq 0, \tag{1}$$

where k, defined for $0 \leq s \leq t < \infty$, with values in the space $\mathbb{R}^{d\times d}$ of $d \times d$ matrices, is a Volterra kernel ($k(t,s) = 0$ when $s > t$) and the forcing function f is defined for $t \geq 0$ and takes values in $\mathbb{R}^d$. We assume that $f(t)$ and $k(t,s)$ are at least continuous for $t \geq 0$ and $0 \leq s \leq t < +\infty$ respectively. Existence and uniqueness results can be found in [10].

The stability analysis of the numerical method used in the simulation of these equations is essential to assure that the approximating solution preserves the characteristics and the long time behavior of the continuous problem.

An analysis of this kind is already well known in the literature (see [1,2,14,13,17] and [4,3] for a thorough discussion), but in spite of the considerable advances that have been made in the last years, there are still many aspects to investigate and open question to answer. The main idea in the numerical stability analysis is to identify more and more realistic model problems (representative of the applications) and to study how numerical methods behave when simulating them.

* Corresponding author.
E-mail addresses: eleonora.messina@unina.it (E. Messina), antonia.vecchio@cnr.it (A. Vecchio).

http://dx.doi.org/10.1016/j.apnum.2017.01.011
0168-9274/

Our approach consists in discretizing some suitable classes of equations (1) by Direct Quadrature (DQ) methods and study the asymptotic behavior of the numerical error as a function of the stepsize h.

In Section 2 we introduce some basic material needed in the paper. In Section 3 we define the model problem and obtain conditions on the stepsizes h for the DQ methods to be stable. Furthermore, we identify subclasses of problems for which the methods turn out to be unconditionally stable. In Section 4 we prove the boundedness of the numerical solution, whenever the forcing term $f(t)$ is bounded. In Section 5 numerical examples are given and Section 6 contains some concluding remarks.

2. Preliminaries

Consider the general DQ–(ρ,σ) method for the numerical integration of the linear VIE (1)

$$y_n = f_n + h\sum_{j=0}^{n_0-1} w_{nj}k_{nj}y_j + h\sum_{j=n_0}^{n} \omega_{n-j}k_{nj}y_j,\ n = n_0, n_0+1, \ldots, \tag{2}$$

where $y_n \simeq y(t_n)$, $f_n = f(t_n)$, $k_{nj} = k(t_n, t_j)$, with $t_n = nh$ for $n = 0, 1, \ldots$, $h > 0$ is the stepsize and w_{nj}, ω_j are the weights of a Gregory quadrature formula (which corresponds to Adams–Moulton (ρ,σ)-quadrature formula, see [4,13,18] for a general discussion on DQ–(ρ,σ) methods). Here, we assume that the weights are non-negative and that $y_0 = f_0, \ldots, y_{n_0-1}$, $n_0 \geq 1$, are given starting values. The property

$$W = \sup_n \omega_n < +\infty \tag{3}$$

of (ρ,σ) weights (see for example [18]) will be useful in the following discussion.

Let us write the error equation for the DQ method (see for example [12, p. 104])

$$e_n = T_n(h) + h\sum_{j=0}^{n_0-1} w_{nj}k_{nj}\eta_j + h\sum_{j=n_0}^{n} \omega_{n-j}k_{nj}e_j,\ n = n_0, n_0+1, \ldots \tag{4}$$

where $e_n = y(t_n) - y_n$ is the global error, $T_n(h)$ represents the local discretization error and η_j, $j = 0, \ldots, n_0 - 1$ are the starting errors. According to [18] the local error $T_n(h)$ has an expansion of order p, that is (see definition in [12, p. 105]) there exists a function $Q(t)$ such that

$$T_n(h) = Q(t_n)h^p + R_n(h)h^q,\ q > p. \tag{5}$$

From now on we assume that $\sup_n \|Q(t_n)\| \leq \bar{Q}$, with $\bar{Q}$ independent of n and h and $\sup_{0<h<h^*} \sup_{n\geq 0} \|R_n(h)\|$ is finite for small $h^* > 0$. A remark on these assumptions is reported at the end of Section 3.

All the analysis below will be based on the following definition of stability of the numerical scheme (2):

Definition 1. The numerical method (2) for solving a class of Volterra integral equations (1) is said to be stable for that class of equations if, using a given stepsize $h > 0$, the global error e_n given by (4)–(5) satisfies

$$\|e_n\| \leq Eh^p,\ \forall n \geq 0,$$

with E constant independent of n and h.

According to Definition 1, our request for a numerical method to be stable is not only that the local errors do not build up catastrophically in the step by step process, but also that the global errors remain bounded for a fixed stepsize $h > 0$, and may be made arbitrary small reducing h, over long-time intervals.

From now on we will assume that h satisfies

$$det(I - h\omega_0 k_{nn}) \neq 0, \tag{6}$$

where I is the identity matrix of size d, in order to assure the solvability of the discrete problem (2). Furthermore, we assume the starting errors η_j, $j = 0, \ldots, n_0 - 1$, to be accurate at least of order p, so to incorporate them into the local error $T_n(h)$ (that is $T_n(h)$ is now equal to $T_n(h) + h\sum_{j=0}^{n_0-1} w_{nj}k_{nj}\eta_j$) and make (5) still valid.

3. Numerical stability for linear equations

This section is devoted to the study of the behavior of a numerical method of the form (2) when applied to problem (1). The class of problems that we are going to examine consists of equations (1) where the kernel k satisfies: