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# Generalized weighted Birkhoff–Young quadratures with the maximal degree of exactness ☆

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Dedicated to Professor Francesco A. Costabile on his 70th birthday

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ABSTRACT

Several types of quadratures of Birkhoff–Young type, as well as a sequence of the weighted generalized quadrature rules and their connection with multiple orthogonal polynomials, are considered. Beside a short account on a recent result on the generalized $(4n+1)$-point Birkhoff–Young quadrature, general weighted quadrature formulas of Birkhoff–Young type with the maximal degree of exactness are given. It includes a characterization and uniqueness of such rules, as well as numerical construction of nodes and weight coefficients. An explicit form of the node polynomial of such kind of quadratures with respect to the generalized Gegenbauer weight function is obtained. Finally, a sequence of generalized quadrature formulas is studied and their node polynomials are interpreted in terms of multiple orthogonal polynomials.



## 1. Introduction and preliminaries

The well-known quadrature formula for numerical integration over the line segment $[z_0-h, z_0+h]$ of analytic functions in the complex domain $\Omega=\{z : |z-z_0|\le r\}$, $|h|\le r$,

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z=\frac{h}{15}\Big\{24f(z_0)+4\big[f(z_0+h)+f(z_0-h)\big]-\big[f(z_0+\mathrm{i}h)+f(z_0-\mathrm{i}h)\big]\Big\}+R_5^{BY}(f)$$

was obtained by Birkhoff and Young [5], and it is exact for all algebraic polynomials of degree at most five. Young [32] proved that its error term can be estimated by

$$|R_5^{BY}(f)|\le\frac{|h|^7}{1890}\max_{z\in S}|f^{(6)}(z)|,$$

where $S$ denotes the square with vertices $z_0+\mathrm{i}^k h$, $k=0,1,2,3$ (see also the monograph [6, p. 136]). Birkhoff–Young rule can be compared with the so-called extended Simpson rule (cf. [28, p. 124]) with the nodes $z_0$, $z_0\pm h$, $z_0\pm 2h$, and

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$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z \approx \frac{h}{90}\Big\{114 f(z_0)+34\Big[f(z_0+h)+f(z_0-h)\Big]-\Big[f(z_0+2h)+f(z_0-2h)\Big]\Big\}+R_5^{ES}(f),$$

where

$$|R_5^{ES}(f)| \sim \frac{|h|^7}{756}|f^{(6)}(\zeta)|, \qquad 0<\frac{\zeta-(z_0-2h)}{4h}<1.$$

Both formulas use $N=5$ points and have the same algebraic degree of exactness $d=5$, but $|R_5^{BY}(f)| \approx 0.4\,|R_5^{ES}(f)|$.

In 1976 Lether [10] transformed Birkhoff–Young formula from $[z_0-h, z_0+h]$ to $[-1,1]$ (of course, without loss of generality),

$$I(f)=\int_{-1}^{1} f(z)\,\mathrm{d}z=\frac{8}{5}f(0)+\frac{4}{15}\big[f(1)+f(-1)\big]-\frac{1}{15}\big[f(\mathrm{i})+f(-\mathrm{i})\big]+R_5(f), \tag{1.1}$$

and pointed out that the three point Gauss–Legendre quadrature which is also exact for all polynomials of degree at most five, is more precise than (1.1) and he recommended it for numerical integration. However, Tošić [29] improved the quadrature (1.1) in a simple way taking its nodes at the points $\pm r$ and $\pm \mathrm{i}r$, with $r\in(0,1)$, instead of $\pm 1$ and $\pm\mathrm{i}$, respectively, and derived an one-parametric family of quadrature rules in the form

$$\begin{aligned} I(f)=2\left(1-\frac{1}{5r^4}\right)f(0)+\left(\frac{1}{6r^2}+\frac{1}{10r^4}\right)\big[f(r)+f(-r)\big] \\ +\left(-\frac{1}{6r^2}+\frac{1}{10r^4}\right)\big[f(\mathrm{i}r)+f(-\mathrm{i}r)\big]+R_5^T(f;r). \end{aligned} \tag{1.2}$$

It is clear that for $r=1$ it reduces to (1.1). However, for $r=\sqrt{3/5}$, the coefficient of $f(\mathrm{i}r)+f(-\mathrm{i}r)$ vanishes, and it reduces to the three point Gauss–Legendre formula,

$$I(f)=\frac{8}{9}f(0)+\frac{5}{9}\left[f\left(\sqrt{\frac{3}{5}}\right)+f\left(-\sqrt{\frac{3}{5}}\right)\right]+R_3^G(f), \tag{1.3}$$

where $R_3^G(f)=R_5^T(f;\sqrt{3/5})$.

Expanding the error-term $R_5^T(f;r)$ in (1.2) in the form

$$R_5^T(f;r)=\left(-\frac{2}{3\cdot 6!}r^4+\frac{2}{7!}\right)f^{(6)}(0)+\left(-\frac{2}{5\cdot 8!}r^4+\frac{2}{9!}\right)f^{(8)}(0)+\cdots, \tag{1.4}$$

and putting $r=\sqrt[4]{3/7}$ in order to vanish the first term in (1.4), Tošić [29] obtained a five-point formula of algebraic degree of exactness seven,

$$\begin{aligned} I(f)=\frac{16}{15}f(0)+\frac{1}{6}\left(\frac{7}{5}+\sqrt{\frac{7}{3}}\right)\left[f\left(\sqrt[4]{\frac{3}{7}}\right)+f\left(-\sqrt[4]{\frac{3}{7}}\right)\right] \\ +\frac{1}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right)\left[f\left(\mathrm{i}\sqrt[4]{\frac{3}{7}}\right)+f\left(-\mathrm{i}\sqrt[4]{\frac{3}{7}}\right)\right]+R_5^{MF}(f), \end{aligned} \tag{1.5}$$

with the error-term

$$R_5^{MF}(f)=R_5^T(f;\sqrt[4]{3/7})=\frac{1}{793800}f^{(8)}(0)+\frac{1}{61122600}f^{(10)}(0)+\cdots\approx 1.26\cdot 10^{-6}f^{(8)}(0).$$

We note that the error term in the Gaussian formula (1.3) is given by

$$R_3^G(f)=R_5^T(f;\sqrt{3/5})=\frac{1}{15750}f^{(6)}(0)-\frac{1}{226800}f^{(8)}(0)+\cdots\approx 6.35\cdot 10^{-5}f^{(6)}(0).$$

Quadrature formulae of Birkhoff–Young type for analytic functions have been investigated in several papers in different directions (cf. [1,15,20,22]). These formulas can also be used to integrate real harmonic functions (see [5]). In addition, we mention also that Lyness and Delves [12] and Lyness and Moler [13], and later Lyness [11], developed formulae for numerical integration and numerical differentiation of complex functions.