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Some error bounds for Gauss–Jacobi quadrature rules

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ABSTRACT

We estimate the error of Gauss–Jacobi quadrature rule applied to a function f, which is supposed locally absolutely continuous in some Besov type spaces, or of bounded variation on $[-1, 1]$. In the first case the error bound concerns the weighted main part φ-modulus of smoothness of f introduced by Z. Ditzian and V. Totik, while in the second case we deal with a Stieltjes integral with respect to f. The stated estimates generalize several error bounds from literature and, in both the cases, they assure the same convergence rate of the error of best polynomial approximation in weighted L^1 space.

1. Introduction

Gauss–Jacobi quadrature rules have been extensively studied in literature (see e.g. [1,4,10,14] and the references therein). For a given $n \in \mathbb{N}$, they provide the following approximation

$$\int_{-1}^{1} f(x)u(x)dx \approx \sum_{k=1}^{n} \lambda_k f(x_k),$$

where $u(x) = v^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, is a given Jacobi weight, $p_n(u, x)$ is the associated orthonormal Jacobi polynomial of degree n and positive leading coefficient,

$$\lambda_k := \left[\sum_{j=0}^{n-1} p_j(u, x_k)^2\right]^{-1}, \qquad k = 1, \dots, n,$$

are the well-known Christoffel numbers, and $x_1 < x_2 < \dots < x_n$ are the zeros of $p_n(u, x)$. We assume that $f \in L^1_u := \{f : \|fu\|_1 := \int_{-1}^{1} |f(x)|u(x)dx < \infty\}$ is defined at these nodes and locally bounded in $[-1, 1]$ (i.e. bounded in each $[a, b] \subseteq (-1, 1)$).

Set

$$R_n(f)_u := \left| \int_{-1}^{1} f(x)u(x)dx - \sum_{k=1}^{n} \lambda_k f(x_k) \right|,$$

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and denote by $\mathbb{P}_n$ the set of all polynomials of degree at most n. It is well known that

$$R_n(f)_u = 0, \qquad \forall f \in \mathbb{P}_{2n-1}, \tag{1}$$

while, in the general case, $R_n(f)_u \to 0$ as $n \to \infty$ and the rate of convergence depends on the smoothness properties of f. Regarding this, various error estimates have been proved by several authors under different smoothness assumptions of the integrand function (see e.g. [5,6,9,12,15–18,25,31]).

In particular, for the non-weighted case (i.e. $u(x) = 1$), De Vore and Scott [5] proved that if for some integer $s \le 2n$ we have $\|f^{(s)}\varphi^s\|_1 < \infty$, being here and in the following $\varphi(x) := \sqrt{1-x^2}$, then we have

$$R_n(f) \le \frac{C_s}{n^s}\|f^{(s)}\varphi^s\|_1, \tag{2}$$

where $R_n(f)$ denotes the quadrature error when $u = 1$ and C_s is a positive constant depending on s, but independent of f and n.

Later on, Ditzian and Totik [7, Section 7.4] combined their results with (2) and stated an error bound based on special moduli of smoothness rather than on the derivatives of f. More precisely, they proved that

$$R_n(f) \le \frac{M_r}{n}\int_0^{\frac{1}{n}} \frac{\omega_\varphi^r(f,t)}{t^2}dt, \qquad n > r, \tag{3}$$

holds for all $f \in L^1$, where $M_r > 0$ is independent of n and f, and

$$\omega_\varphi^r(f,t) := \sup_{0<h\le t} \|\Delta_{h\varphi}^r f\|_1,$$

being $\Delta_{h\varphi}^r f$ the central rth difference of f of variable step size $h\varphi(x)$.

Inspired by Ditzian–Totik results, in Section 2 (Theorem 1) we state the following estimate

$$R_n(f)_u \le \frac{\mathcal{C}}{n}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^r(f,t)_u}{t^2}dt, \qquad n > r, \qquad \mathcal{C} \ne \mathcal{C}(n,f), \tag{4}$$

where [7, eq. (8.1.2)]

$$\Omega_\varphi^r(f,t)_u := \sup_{0<h\le t} \|(\Delta_{h\varphi}^r f)u\|_{L^1[-1+2r^2h^2,\,1-2r^2h^2]},$$

throughout the paper $\mathcal{C}$ denotes a positive constant which can take different values in different formulas, and $\mathcal{C} \ne \mathcal{C}(n,f)$ means that $\mathcal{C}$ does not depend on n and f.

Comparing with (3), we observe that (4) concerns the more general weighted case and, instead of the complete modulus ω_φ^r, it regards the so-called main part modulus Ω_φ^r. We recall that, in the weighted case, $\omega_\varphi^r(f,t)_u$ is defined only for Jacobi weights $u = v^{\alpha,\beta}$ with $\alpha, \beta \ge 0$ (see [7, Remark 6.1.2]), while in (4) $\Omega_\varphi^r(f,t)_u$ results defined without any restriction on $u \in L^1$ and it is often easier to compute, since it does not take into account of the function values close to the extremes ± 1 (see [7, Section 8.5] for some examples and calculations).

Moreover, the complete and main part moduli are related by [7, Theorem 6.2.2]

$$\mathcal{C}^{-1}\Omega_\varphi^r(f,t)_u \le \omega_\varphi^r(f,t)_u \le \mathcal{C}\int_0^t \frac{\Omega_\varphi^r(f,\tau)_u}{\tau}d\tau, \qquad \mathcal{C} \ne \mathcal{C}(f,t),$$

and they both well characterize the rate of convergence to zero of the error of best polynomial approximation

$$E_n(f)_u := \inf_{P\in\mathbb{P}_n} \|(f-P)u\|_1, \tag{5}$$

in terms of the smoothness of f. In particular, we have [7, Corollary 8.2.2]

$$E_n(f)_u = \mathcal{O}(n^{-a}) \Longleftrightarrow \Omega_\varphi^r(f,t)_u = \mathcal{O}(t^a), \qquad r > a > 0. \tag{6}$$

Hence, from (4) we deduce that if $E_n(f)_u = \mathcal{O}(n^{-a})$ holds for any $a > 1$, then $R_n(f)_u = \mathcal{O}(n^{-a})$ holds too.

Indeed this result was already known in Sobolev spaces

$$W_u^s = \{f \in L_u^1 : \|f^{(s)}\varphi^s u\|_1 < \infty\}, \qquad s \in \mathbb{N}, \tag{7}$$

where $E_n(f)_u = \mathcal{O}(n^{-s})$ holds [23] and $R_n(f)_u = \mathcal{O}(n^{-s})$ follows from the weighted version of (2) (namely (16), which we deduced from (4), but it was already proved [19, Th. 5.1.8]).

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