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Unconditional error estimates for time dependent viscoelastic fluid flow ☆

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ABSTRACT

The unconditional convergence of finite element method for two-dimensional time-dependent viscoelastic flow with an Oldroyd B constitutive equation is given in this paper, while all previous works require certain time-step restrictions. The approximation is stabilized by using the Discontinuous Galerkin (DG) approximation for the constitutive equation. The analysis bases on a splitting of the error into two parts: the error from the time discretization of the PDEs and the error from the finite element approximation of corresponding iterated time-discrete PDEs. The approach used in this paper can be applied to more general couple nonlinear parabolic and hyperbolic systems.

1. Introduction

The time dependent Oldroyd B model for viscoelastic flow is governed in general by the following system of equations:

$$Re\mathbf{u}_t + \nabla p - 2(1-\alpha)\nabla\cdot D(\mathbf{u}) - \nabla\cdot\sigma = \mathbf{f}, \quad \text{in } \Omega, \tag{1}$$

$$\lambda\sigma_t + \sigma + \lambda(\mathbf{u}\cdot\nabla)\sigma + \lambda g_a(\sigma,\nabla\mathbf{u}) - 2\alpha D(\mathbf{u}) = 0, \quad \text{in } \Omega, \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega, \tag{3}$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega, \tag{4}$$

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \text{in } \Omega, \tag{5}$$

$$\sigma(0,x) = \sigma_0(x), \quad \text{in } \Omega, \tag{6}$$

where Re is the Reynolds number, λ is the Weissenberg number, $0<\alpha<1$ is considered as the fraction of viscoelastic viscosity, and $\mathbf{f}$ the body forces, the fluid velocity vector $\mathbf{u} = (u_1, u_2)$, the pressure p, and the stress σ, which is the

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viscoelastic part of the total stress tensor $\sigma_{tot} = \sigma + 2(1-\alpha)D(\mathbf{u}) - p\mathbf{I}$. Besides, $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the rate of the strain tensor and $g_a(\sigma, \nabla\mathbf{u})$ is defined by

$$g_a(\sigma, \nabla\mathbf{u}) = \frac{1-a}{2}(\sigma\nabla\mathbf{u} + (\nabla\mathbf{u})^T\sigma) - \frac{1+a}{2}((\nabla\mathbf{u})\sigma + \sigma(\nabla\mathbf{u})^T) \tag{7}$$

with $a \in [-1, 1]$. The viscoelastic flows are important to the understanding of many problems in Non-Newtonian fluid mechanics, particularity those related to flow instabilities [15,23,27]. The underlying equations are usually considered as the (parabolic) conservation of momentum and incompressibility equations for fluid flow, coupled with a (hyperbolic) constitutive equation for the viscoelastic component of the stress. Some existence results for the viscoelastic flows with a differential constitutive law is obtained by Guillope and Saut [12], more complete discussion of existence and uniqueness issues can be found in [24].

To avoid the introduction of spurious oscillations in finite element approximation for the constitutive equation, two ways are generally used: the discontinuous Galerkin (DG) approximation [3,21] or the Streamline Upwind Petrov Galerkin (SUPG) approximation [25].

For the viscoelastic flows, in [9,10], Ervin and Miles analyzed an implicit Euler time discretization and a SUPG discretization for the constitutive equation, which also required some conditions $k,\ \nu = O(h^{d/2})$, ν is the stabilization parameter of SUPG method, $d = 2, 3$ corresponding to $2D$ or $3D$ model. Error analysis of a modified Euler–SUPG approximation to time dependent viscoelastic flow problem was presented by Bensaada and Esselaoui in [5], with a weaker discretization constraint $k = O(h)$. Chrispell, Ervin and Jenkins described and analyzed a fractional step θ-method for the time-dependent problem in [7], however, required $k = O(h^2)$. The reader may find more relevant work, for example, [6,22] and the references therein.

DG methods have become popular due to their computational flexibility and their ability to incorporate physical properties. In 1986, Johnson and Pitkäranta [14] analyze the DG method for a scalar hyperbolic equation. Later, Atkins and Shu gave quadrature-free implementation of DG method for hyperbolic equations [2]. In [4], finite element error analysis to time dependent viscoelastic flow was first analyzed by Baranger and Wardi, using the implicit Euler temporal discretization and DG approximation for the hyperbolic constitutive equation, required certain time-step conditions $k = O(h^{3/2})$. Zhang and his coauthors [28] extend this method with defect correction technique at high Weissenberg numbers. Later, Ervin and Heuer proposed a Crank–Nicolson time discretization scheme with a DG approximation for the constitutive equation, and improved the time-step conditions by $k = O(h^{d/4})$ and yielded a first-order temporal approximation for the pressure in [8].

With some projection operators R_h, a key issue to bound the numerical solution in L^∞ norm in a traditional way by the mathematical induction with an inverse inequality, such as

$$||\sigma^{h,n}||_\infty \le ||R_h\sigma^n||_\infty + ||R_h\sigma^n - \sigma^{h,n}||_\infty \le C + h^{-d/2}(Ch^m + k^l),$$

which immediately leads to a time step-size restriction $k^l \le Ch^m$, here, for example, usually, $l = 1$ for Back–Euler method and $l = 2$ for Crank–Nicolson method, m depends on the finite element spaces. The restriction could become more serious while in a high-dimensional space or with a non-uniform mesh, since a very small time-step may be used and extremely time-consuming in practical computations.

The paper is concerned with the unconditional optimal error estimates of finite element approximation to the viscoelastic flows, with DG discretization for the constitutive equation. First, we will introduce a corresponding iterated time-discrete system, then, we split the finite element error into two parts, the error in the temporal direction and the error in the spatial direction. We do not need any time-step condition to bound the numerical solutions. Rigorous analysis of the regularity of the solution to the iterated time-discrete PDEs is a key to our approach. Such ideas are first developed by Li and Sun [18,19] to analyze Galerkin finite element methods for some nonlinear parabolic equations, then, applied to nonlinear Thermistor equations [20], characteristics-mixed FEMs for miscible displacement in porous media [26], Landau–Lifshitz Equation [1] and so on.

The rest of the paper is organized as follows. In section 2, we introduce the implicit Euler time discretization and a FE discretization for the time-dependent viscoelastic flows with the constitutive equation stabilized by DG approximation, we also present our main results. In section 3, we introduce an iterated linearized time-discrete system and present rigorous analysis for the regularity of the solution of this system. Optimal error estimates are proved unconditionally with the regularity in section 4. Finally, a summary and discussion of continuing work are presented in section 5.

2. Implicit Euler time-discretization and FE discretization for the time-dependent viscoelastic flows

We introduce some notations first. The $L^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$, and the $L^p(\Omega)$ norm by $\|\cdot\|_{L^p}$, with the special cases of $L^2(\Omega)$ and $L^\infty(\Omega)$ norms being written as $\|\cdot\|$ and $\|\cdot\|_\infty$. For $k \in N$, we denote the norm associated with the Sobolev space $W^{m,p}(\Omega)$ by $\|\cdot\|_{W^{m,p}}$, with the special case $W^{m,2}(\Omega)$ being written as $H^m(\Omega)$ with the norm $\|\cdot\|_m$ and seminorm $|\cdot|_m$. In order to introduce a variational formulation, we set the spaces X, Q, S, V as follows.

$$X := H_0^1(\Omega)^2 := \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v} = \mathbf{0} \text{ on } \Gamma\},$$

$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega);\ \int_\Omega q\,dx = 0\},$$