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A second-order hybrid finite volume method for solving the Stokes equation

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ABSTRACT

This paper presents a second-order hybrid finite volume method for solving the Stokes equation on a two dimensional domain. The trial function space of the method for velocity is chosen to be a quadratic conforming finite element space with a hierarchical decomposition technique on triangular meshes, and its corresponding test function space consists of piecewise constant functions and piecewise quadratic polynomial functions based on a dual partition of the domain. The trial function space and test function space of the method for pressure are chosen to be a linear finite element space. We derive the inf-sup conditions of the discrete systems of the method on triangular meshes by using a relationship between the finite volume method and the finite element method. The well-posedness of the proposed finite volume method is obtained by using the Babuska–Lax–Milgram theorem. The error estimates of the optimal order are obtained in the H^1-norm for velocity and in the L^2-norm for pressure. Numerical experiments are presented to illustrate the theoretical results.

1. Introduction

We consider in this paper the numerical solution of the Dirichlet problem of the Stokes equation by using a finite volume method. The Dirichlet problem of the Stokes equation is important in fluid mechanics, which has a wide range of practical applications.

There are extensive research ideas developed to improve the numerical approximation of the Stokes equation. Numerical solutions of the Stokes equation are usually obtained through the use of finite element methods (FEMs), for which the reader is referred to [1,2,4,6,16,17] and the references cited therein. This is because FEMs can handle a complex geometry region and is easier to construct higher-order accuracy formats. However, the main drawback of FEMs is its computational complexity and loss of the local conservation property. The conservation of mass is an important criterion in the numerical solutions of computational fluid dynamics. Finite volume methods (FVMs) not only inherit the advantages of FEMs but also require less computational costs and preserve the local conservativeness for the numerical fluxes, which can be fundamen-

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tally important for simulations of many physical models. Thus FVMs are widely used in computational fluid mechanics and other applications [14,18,23,28].

Many FVMs were developed for solving the Stokes equation in recent years. However most of the existing FVMs are limited to the cases where the velocity and pressure fields are approximated by piecewise constants or piecewise linear polynomials [10,11,13,15,19,24,25,27,29,31–33]. There is almost no literature available for FVMs using quadratic functions or higher-order for the velocity term on triangle meshes for the Stokes problem. The theoretical analysis on higher-order FVMs for the Stokes problem on the quadrilateral meshes remains unknown. It is worth mentioning that the high order FVMs researches on elliptic equations had great breakthroughs in recent years. For example, some convergent properties of both linear and quadratic finite volume methods for elliptic equations were analyzed in [30]. A class of high order hybrid FVMs for elliptic equations was presented in [5] by combining high order FEMs and linear FVMs. The necessary and sufficient conditions for the uniform ellipticity of the family of the bilinear forms of the FVMs for elliptic problems were established in terms of geometric requirements on triangle meshes [7–9], especially paper [8] supplies a general construction of the test spaces which match the trial spaces for the FVMs. The biquadratic FVMs for elliptic equations were established on quadrilateral meshes [20]. A proof of the inf-sup condition on vertex-centered FVMs of arbitrary order for elliptic equations was established on quadrilateral meshes [34,35].

The main purpose of this paper is to develop a second-order hybrid FVM for solving the Stokes problem with a hybrid quadratic Lagrange finite volume element for the velocity term and a linear finite element for the pressure term of the equation. The hybrid FVMs offer several advantages. Besides the hybrid FVMs have the advantages of the general FVMs, the most advantage is that the dual grid partitions of the hybrid element are simple. The dual grids for higher-order hybrid FVMs can be the same as that for linear ones [5,7,8], while the traditional way is to introduce a control volume for each node [18,21], in which the geometry of the control volumes will complicate the analysis and implementation of higher-order FVMs. For example, the dual grids for the Lagrange quadratic finite volume elements consist of the control volumes of vertices and middle points of edges, while for the hybrid Lagrange quadratic finite volume elements it only needs the control volumes of vertices. Thus, the hybrid FVMs can simplify its theoretical analysis. In addition, for the second discrete bilinear form of the FVM for solving the Stokes problem in this paper, we can get the inf-sup condition of the discrete bilinear form from the existing FEMs results. In fact, the main task in the analysis for FVMs and FEMs of the Stokes equation is to verify the inf-sup conditions for the discrete systems.

This paper is organized in five sections. In the next section, we introduce the stationary Stokes equation and establish the second-order hybrid FVM for solving the Stokes problem. The continuity and inf-sup conditions of the discrete bilinear forms that result from the FVM are established in the third section, and the solvability of the proposed FVM is given. Optimal order error estimates for the FVM are established in the fourth section. In Section 5 we present numerical examples to illustrate the error estimates and the convergence orders.

2. A hybrid FVM for the Stokes problem

We establish in this section the hybrid FVM for solving the Stokes equation.

Let Ω denote a domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. We shall use the standard notation $\mathrm{L}^2(\Omega)$ for the space of square Lebesgue integrable functions with the norm $\|\cdot\|_{0,\Omega}$ and $\mathrm{H}^m(\Omega)$ for the Sobolev function space with the norm $\|\cdot\|_{m,\Omega}$ and the semi-norm $|\cdot|_{m,\Omega}$. Generally, we shall drop the subscript Ω for brevity. We introduce the following spaces

$$\mathrm{L}_0^2(\Omega) := \left\{ p : p \in \mathrm{L}^2(\Omega), \int_\Omega p = 0 \right\},$$

$$\mathrm{H}_0^1(\Omega) := \left\{ v : v \in \mathrm{H}^1(\Omega), v|_{\partial\Omega} = 0 \right\},$$

and

$$\mathbf{H^m}(\Omega) := (\mathrm{H}^m(\Omega))^2.$$

Consider the Stokes problem for steady flow of a heavily viscous fluid with Dirichlet boundary conditions: Find $\mathbf{u} := (u_1, u_2) \in \mathbf{H^2}(\Omega)$ and $p \in \mathrm{H}^1(\Omega)$ such that

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{f}$ is the given external volumetric force acting on the fluid, $\mu > 0$ is the given constant fluid viscosity, $\mathbf{u}$ is the vectorial fluid velocity field to be determined and p is the scalar pressure to be determined. Note that if $(\mathbf{u}, p)$ solves (1) then $(\mathbf{u}, p + c)$ also solves (1) for any constant c. That is the pressure p is determined up to a constant. To obtain a unique pressure, we impose an extra condition $p \in \mathrm{L}_0^2(\Omega)$ in the Stokes problem (1). For the sake of simplicity, we assume $\mu := 1$.

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