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The discontinuous Galerkin finite element method for fractional cable equation ☆

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ABSTRACT

The cable equation as one of the best models for simulating neurodynamics can be derived from the Nernst–Planck equation which simulates the electrodiffusion of ions. Recently, some researchers find that in nerve cells molecular diffusion is anomalous subdiffusion. It is much more effective using fractional cable equation for simulating the dynamic behavior. In this paper, by introducing an auxiliary function $w = \partial u/\partial t$, the fractional cable equation can be changed into a system of integro-differential equations. Then a full discrete numerical method for solving the system is studied, where in time axis the discontinuous Galerkin finite element method is used and in spacial axis the Galerkin finite element scheme is adopted. The existence and uniqueness of the numerical solution are included. The convergence is also discussed in detail. Numerical examples are also included to demonstrate the effectiveness of the theoretical results.

1. Introduction

For simulating the electro-diffusion movement of the ions flux in the dendrites, the cable equation is obtained, see [10,23] etc. However, many researchers found that the anomalous diffusion phenomenon in the dendrites occurs more and more frequently, where the time fractional derivative is introduced into the cable equation for modeling this type of diffusion [8,11,25]. Nowadays, fractional cable equation has already become one of the most important tools for simulating the neuronal dynamics phenomenon.

In essence, the fractional cable equation is scoped to fractional diffusion equation and its analytical solution is hard to get. Some numerical methods for solving the fractional diffusion equation have been proposed. One is adopting finite difference ideas both in time and space axes, see [9,14,28] etc. Second is applying finite difference scheme in time axis and using the spectral method along space axis see e.g. [1,17] etc. But these methods are often unstable under some conditions. Sweilama et al. [28] mentioned the unstableness, Abualnaja [1], Karatay and Kale [9] discussed the stable condition respectively. The Galerkin finite element method is an effective and stable method for numerical solving the differential equation. And the finite element method along spacial direction is adopted to numerical solving the fractional differential equation see e.g. [5,6,15,16,18,32–34,36] etc. Li et al. [16] and Liu et al. [18] analyzed the stability of space semi-discrete finite element

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methods. But the approximations along time direction still applied the finite difference methods and the stability of these fully discrete difference/finite element schemes is worthy of concern.

The discontinuous Galerkin finite element method was first proposed and analyzed in the early 1970s as a technique to seek numerical solutions for partial differential equations. Because of its flexibility and efficiency in terms of mesh and shape functions, the discontinuous Galerkin finite element method becomes a very attractive tool for many types of classical differential equations, e.g. [3,12,13,26,29] and so on. In recent research, the discontinuous Galerkin finite element method is tried to solve the fractional differential equations. Deng and Hesthaven [4], Xu and Hesthaven [30] proposed the discontinuous Galerkin method in spacial axis for fractional convection–diffusion equations and they all proved that the space semi-discrete discontinuous finite element scheme is stable. But in time axis, the fourth order low storage explicit Runge–Kutta method is introduced, so the stability of fully discrete schemes might be also considered.

Mustapha and Mclean [20], Zheng et al. [35] constructed a numerical approximate scheme for the fractional diffusion-wave and Fokker–Planck equation, for which the discontinuous finite element scheme was adopted in time axis and the Galerkin finite element method was used in space axis. Mustapha etc. [20] discussed the stability of time discontinuous finite element semi-discrete scheme.

There are many other ways to approximate the solution of fractional differential equation, such as Variational iteration method [7,31]; decomposition method [19,27]; wavelet operational method [21,22,24] and so on.

This paper follows the idea in an attempt to generalize the discontinuous Galerkin finite element/Galerkin finite element method to the fractional cable equation. We mainly discuss the fractional cable equation with initial–boundary value described as follows

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = {}_C D_{0,t}^{1-\alpha} \Delta u(x,t) - {}_C D_{0,t}^{1-\beta} u(x,t) + f(u,x,t), t \in (0,T], x \in \Omega, \\ u(x,0) = u_0(x), \qquad x \in \Omega, \\ u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in [0,T], \end{cases} \tag{1}$$

where $0 < \alpha, \beta < 1$, and ${}_C D_{0,t}^{\gamma}$ denotes left Caputo derivative respect to time variable t which will be defined later on. For function f, we need the following supposition: there exist two positive constants m and M, satisfying

$$0 < m \le f(u,x,t) \le M.$$

And we always assume that the following mild Lipschitz continuity conditions on f hold: there exists a positive constant L such that for $u_1, u_2 \in (0,T] \times \Omega$,

$$\|f(u_1,x,t) - f(u_2,x,t)\| \le L\|u_1 - u_2\|.$$

Then we apply the discontinuous Galerkin finite element method in time axis and the Galerkin finite element method in spacial axis to approximate above fractional cable equation.

The rest of this paper is constructed as follows. The fractional derivative space is introduced in Section 2. In Section 3 the discontinuous Galerkin finite element scheme is given. The existence and the uniqueness are also proved in Section 3. The error estimates of the discontinuous Galerkin finite element approximation for the fractional cable equation are studied in Section 4. Numerical experiments are constructed to demonstrate the performance of the proposed scheme in Section 5.

2. Fractional derivatives and related properties

In this section, we will introduce the fractional integrals, Caputo derivatives and their related properties, which will be used in the following finite element analysis.

Definition 1. The α-th order left and right Riemann–Liouville integrals of function $u(x)$ are defined as follows

$$D_{a,x}^{-\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{u(s)}{(x-s)^{1-\alpha}} \mathrm{d}s,$$

$$D_{x,b}^{-\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{u(s)}{(s-x)^{1-\alpha}} \mathrm{d}s,$$

where $\alpha > 0$ and $a < x < b$.

Definition 2. The α-th order left and right Caputo derivatives of function $u(x)$ are defined as follows

$${}_C D_{a,x}^{\alpha} u(x) = D_{a,x}^{-(n-\alpha)} D^n u(x),$$

$${}_C D_{x,b}^{\alpha} u(x) = D_{x,b}^{-(n-\alpha)} (-D)^n u(x),$$

where $n - 1 < \alpha < n \in Z^+$ and $a < x < b$.