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The fictitious domain method with penalty for the parabolic problem in moving-boundary domain: The error estimate of penalty and the finite element approximation ☆

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ABSTRACT

We consider the fictitious domain method with penalty for the parabolic problem in a moving-boundary domain. Two types of penalty (the H^1 and L^2-penalty methods) are investigated, for which we obtain the error estimate of penalty. Moreover, for H^1-penalty method, the H^2-regularity and a-priori estimate depending on the penalty parameter ε are obtained. We apply the finite element method to the H^1-penalty problem, and obtain the stability and error estimate for the numerical solution. The theoretical results are confirmed by the numerical experiments.

1. Introduction

When applying the numerical methods to the partial differential equations in a time-dependent moving-boundary domain, we have to update the mesh to fit the boundary at every time step, which is very time-consuming for computation. And for the domain with complex shape, the finite difference method is not easy to implement.

The fictitious domain method is proposed to tackle these problems, which is to reformulate the original problem to a new problem in a larger, simple-shaped domain (called the fictitious domain) containing the original domain, whose solution approximates to the solution of the original problem. Then, instead of solving the original problem, we introduce a uniform mesh independent of the original boundary to the fictitious domain, and apply the numerical methods to solve the reformulated problem. If we set the fictitious domain large enough such that it contains the original domain in the whole time interval, then there is no need to update the mesh at each time step. Moreover, if we choose the fictitious domain to be a rectangle, then the rectangular/Cartesian mesh can be applied, and the implementation of the finite element/difference/volume method becomes simple.

Actually, the fictitious domain method is successfully applied in numerical simulation for the real-world problem, for example, a blood flow and fluid–structure interactions in thoracic aorta [23] and the spilled oil on coastal ecosystems [22].

There exist several methods for the reformulation. We restrict ourselves to the fictitious domain method with H^1-penalty and L^2-penalty (cf. [2,18,20,19,28,27,29,30]) because of their wide applicability. For the fictitious domain method with Lagrange multiplier, we refer the readers to [7–10].

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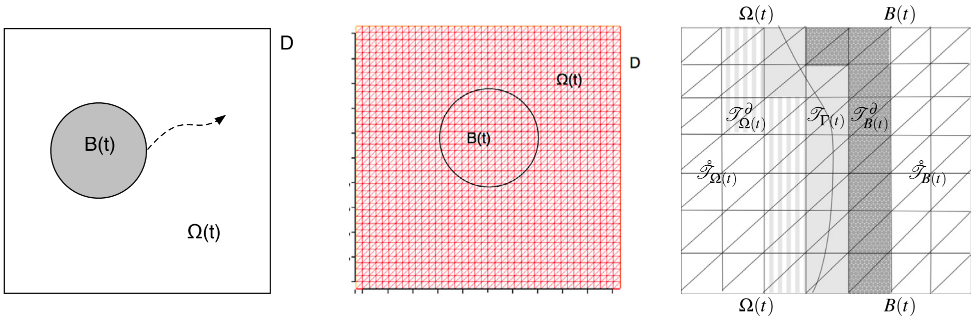

Fig. 1. $\Omega(t)$, $B(t)$, $\Omega(t) \cup B(t) \equiv D$ (left). An example of the uniform triangulation (middle). A decomposition of triangulation $\mathcal{T}_h$ (right), where the left part domain $\Omega(t)$ and the right part domain $B(t)$ are separated by an interface boundary $\Gamma(t)$ (the curve in the middle of graph), and the triangulation is divided into five parts: $\mathring{\mathcal{T}}_{\Omega(t)}$, $\mathcal{T}^{\partial}_{\Omega(t)}$, $\mathcal{T}_{\Gamma(t)}$, $\mathcal{T}^{\partial}_{B(t)}$ and $\mathring{\mathcal{T}}_{B(t)}$. This decomposition is used in Lemma 4.4.

To be more specific, let $Q := \{(x,t) \mid x \in \Omega(t),\ t \in (0,\infty)\}$ be a smooth domain in $\mathbb{R}^d \times (0,\infty)$ with $d = 2, 3$, where $\Omega(t)$ is a smooth and bounded domain in $\mathbb{R}^d$ depending on t. For $T \in (0,\infty)$, we set the domain $Q_T := \{(x,t) \mid x \in \Omega(t),$ $t \in (0,T)\}$ with boundary $\Sigma_T := \{(x,t) \mid x \in \partial\Omega(t),\ t \in (0,T)\}$, and consider the parabolic problem:

$$(\mathbf{P}) \quad \begin{cases} u_t - \Delta u = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u_0 & \text{in } \Omega(0), \end{cases} \tag{1.1}$$

where $f \in L^2(Q_T)$ and $u_0 \in H^1_0(\Omega(0))$.

Suppose there is an obstacle $B(t) \subset \mathbb{R}^d$ surrounded by domain $\Omega(t)$ such that $\overline{\Omega(t) \cup B(t)} \equiv D$ and $\partial\Omega(t) = \partial B(t) \cup \partial D$ for all $t \in [0,T]$, where D is a bounded domain independent of t (see Fig. 1(left)). We set $B_T := \{(x,t) \mid x \in B(t),\ t \in (0,T)\}$, and assume that $B(t)$ is strictly contained in D satisfying $\max_{t\in[0,T]} \text{dist}(\partial D, B(t)) \geq C > 0$, where $\text{dist}(\partial D, B(t))$ means the distance from $B(t)$ to ∂D. To solve (1.1) by finite element method, we have to update the mesh to fit $\Omega(t)$ for each time-step in numerical computation, which is very time-consuming, especially for 3-dimensional problem. Instead of that, we consider a penalty problem in domain $D_T := D \times (0,T)$, where the solution of the penalty problem approximates to the solution u in Q_T. Since D is independent of t, we can introduce the uniform mesh to D, independent of the moving-boundary $\partial B(t)$ (see Fig. 1(middle)), and solve the penalty problem numerically instead of the original problem (**P**).

In the present paper, we consider two penalty methods: the H^1-penalty and L^2-penalty. We introduce the penalty parameter ε with

$$0 < \varepsilon \ll 1,$$

and the characteristic function

$$1_\omega = \begin{cases} 1 & \text{in } \omega, \\ 0 & \text{in } D \backslash \overline{\omega}, \end{cases}$$

where ω is a subdomain in D. We set the notations:

$$\Gamma(t) = \partial B(t), \quad \partial\Omega(t) = \Gamma(t) \cup \partial D, \quad \Sigma_{B,T} = \{(x,t) \mid x \in \Gamma(t),\ t \in (0,T)\},$$

$$\Sigma_{D,T} = \partial D \times (0,T), \quad \Sigma_T = \Sigma_{B,T} \cup \Sigma_{D,T},$$

and extend $f(t)$ and u_0 to D by zero for all $t \in [0,T]$ (i.e. $f(t)|_{B(t)} = 0$, $u_0|_{B(0)} = 0$).

The fictitious domain method with H^1-penalty is to approximate (**P**) by the penalty problem:

$$(\mathbf{P}_\varepsilon \mathbf{H}^1) \quad \begin{cases} u_{\varepsilon t} - \Delta u_\varepsilon = f \quad \text{in } Q_T, \quad u_{\varepsilon t} - \Delta u_\varepsilon + u_\varepsilon = 0 \quad \text{in } B_T, \\ u_\varepsilon|_{Q_T} = u_\varepsilon|_{B_T}, \quad (\nabla u_\varepsilon \cdot n)|_{Q_T} = \dfrac{1}{\varepsilon}\,(\nabla u_\varepsilon \cdot n)|_{B_T} \quad \text{on } \Sigma_{B,T}, \\ u_\varepsilon = 0 \quad \text{on } \Sigma_{D,T}, \quad u_\varepsilon(x,0) = u_0 \quad \text{in } D, \end{cases} \tag{1.2}$$

where $u_\varepsilon|_\omega$ means the restriction of u_ε in ω, and $n(t)$ denotes the unit normal vector to $\Gamma(t)$ towards $B(t)$. The interface boundary condition $(1.2)_2$ on Σ_T implies that $u_\varepsilon(t)$ is H^1-smooth in D but with discontinuous flux on $\Sigma_{B,T}$.

The fictitious domain method with L^2-penalty is to approximate (**P**) by the penalty problem:

$$(\mathbf{P}_\varepsilon \mathbf{L}^2) \quad \begin{cases} u_{\varepsilon t} - \Delta u_\varepsilon + \dfrac{1}{\varepsilon} 1_{B(t)} u_\varepsilon = f & \text{in } D_T, \\ u_\varepsilon = 0 & \text{on } \Sigma_{D,T}, \\ u_\varepsilon(x,0) = u_0 & \text{in } D. \end{cases} \tag{1.3}$$