Applied Numerical Mathematics 115 (2017) 68–81

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Piece-wise moving least squares approximation ☆

Wen Li [a,b], Guohui Song [b], Guangming Yao [b,*]

[a] School of Mathematics, Taiyuan University of Technology, 030024, China
[b] Department of Mathematics, Clarkson University, Potsdam, NY 13699, USA

ARTICLE INFO

Article history:
Received 13 September 2016
Received in revised form 31 December 2016
Accepted 4 January 2017
Available online 10 January 2017

Keywords:
Moving least square
RBF
PDE
Interpolation

ABSTRACT

The standard moving least squares (MLS) method might have an expensive computational cost when the number of test points and the dimension of the approximation space are large. To reduce the computational cost, this paper proposes a piece-wise moving least squares approximation method (PMLS) for scattered data approximation. We further apply the PMLS method to solve time-dependent partial differential equations (PDE) numerically. It is proven that the PMLS method is an optimal design with certain localized information. Numerical experiments are presented to demonstrate the efficiency and accuracy of the PMLS method in comparison with the standard MLS method in terms of accuracy and efficiency.

1. Introduction

The moving least squares (MLS) [21] is a popular method of approximating a function from a set of its values at some scattered data points. It is a flexible meshless method that does not require the construction of a mesh on the domain. It has been widely used in curve and surface fitting [7,21,27,32], and many meshless weak-form methods for solving PDEs, such as the diffuse element method (DEM) [29], the element-free Galerkin methods (EFG) [5] and the meshless local Petrov–Galerkin (MLPG) approach [3], etc.

For each test point, the approximation function of the MLS method is assumed to lie in a finite-dimensional approximation space with certain basis and its coefficients are calculated through a weighted least-squares problem with the weights concentrating at the region around the point. We point out that the weight is depending (moving) on the test point. That is, for different test points, we need to solve different least square problems with the size given by the dimension of the approximation space. The MLS might have a very expensive computational cost when the number of the test points and the dimension of the approximation space are large.

To reduce the computational cost of the standard MLS, many techniques have been introduced in the literature [23]. We propose in this paper a piece-wise moving least squares (PMLS) method. The standard MLS considers a "point-wise" weight that is different for each test point. We will use a "piece-wise" weight instead in the PMLS method. Specifically, we would decompose the domain into some small and disjoint regions and define the weight function for each region rather than for each test point. It is in particular useful in reducing the computational cost when there are many test points lying in the same region.

Moreover, we will consider the PMLS in the view of optimal recovery. In particular, we shall show that it is an optimal design with certain localized information. For the approximation spaces in our numerical experiments, we will focus on two

☆ This work is supported in part by grant NSF-DMS 1521661.
* Corresponding author.
E-mail addresses: liw2@clarkson.edu (W. Li), gsong@clarkson.edu (G. Song), gyao@clarkson.edu (G. Yao).

http://dx.doi.org/10.1016/j.apnum.2017.01.001
0168-9274/Published by Elsevier B.V. on behalf of IMACS.

different basis functions: polynomials and radial basis functions (RBF). The use of RBFs in engineering and sciences leads to many advantages in terms of simplification in high-dimensional problems [8,10,12]. We will test the performance of the proposed PMLS method in scattered data approximation of some benchmark test functions.

We will further apply the PMLS method in numerical solutions of some time-dependent PDEs. There are generally two kinds of approaches for solving time-dependent PDEs numerically [30]. One way is to convert the PDE to a system of ordinary differential equations (ODE) [1,2]. Many traditional techniques of solving system of ODEs could be employed afterwards. The other approach is to apply time discretization and spatial discretization respectively [4,6,16]. Two approaches are basically the same except that the first approach transfers a PDE into a system of ODEs by discretization of the spatial domain first. This is in contrary to the second approach that usually discretizes the time domain first and then discretizes the spatial domain afterwards. A series of PDEs usually needs to be solved in the second approach. We will take the latter approach in this paper. When an explicit time stepping method is used, the time dependent PDE becomes a fitting problem of the solution to the PDEs at the previous time step and approximation of its derivatives. Specifically, we will use the traditional forward Euler formula to discretize the time domain and then apply the PMLS method in spatial discretization of each time step.

The rest of this paper is organized as follows. In section 2, we introduce the piece-wise moving least squares (PMLS) method for scattered data approximation. We prove that the PMLS method is an optimal design with certain localized information in Section 3. We implement in Section 4 the PMLS method in both scattered data approximation and numerical solution of time-dependent PDEs. In particular, we compare the accuracy and efficiency of the proposed PMLS method with the standard MLS method there.

2. Piece-wise moving least squares

We will introduce the PMLS method for scattered data approximation in this section. To this end, we first review the standard MLS method and some notations.

We first give a brief description of the standard MLS as described in [13]. Suppose $\Omega \subseteq \mathbb{R}^d$ is the domain of an unknown function f and $\boldsymbol{x}_j \in \Omega$, $1 \le j \le N$, are some scattered data points in the domain. We are given function values $\boldsymbol{f} = (f(\boldsymbol{x}_j) : 1 \le j \le N)$ on such data points. Assume the approximation space $\mathcal{U}$ is a finite-dimensional space with basis $\{u_1, u_2, \ldots, u_m\}$. For any $\boldsymbol{x} \in \Omega$ and a weight function $w : \mathbb{R} \to \mathbb{R}$, we define a weighted ℓ^2 inner product for functions g_1, g_2 on Ω

$$\langle g_1, g_2 \rangle_{w_{\boldsymbol{x}}} = \sum_{i=1}^{N} g_1(\boldsymbol{x}_i) g_2(\boldsymbol{x}_i) w(\|\boldsymbol{x}_i - \boldsymbol{x}\|),$$

where $\|\cdot\|$ is the Euclidean norm. For a test point $\boldsymbol{y} \in \Omega$, we will try to find the best approximation function $T_{\boldsymbol{y}}$ in the approximation space $\mathcal{U} = \operatorname{span}\{u_1, \ldots, u_m\}$ such that it is "close" to f with respect to the norm $\|\cdot\|_{w_{\boldsymbol{y}}}$ induced by the above weighted inner product $\langle \cdot, \cdot \rangle_{w_{\boldsymbol{y}}}$. That is, we define

$$T_{\boldsymbol{y}} = \operatorname{argmin}_{g \in \mathcal{U}} \|g - f\|_{w_{\boldsymbol{y}}}. \tag{2.1}$$

We next give a reformulation of $T_{\boldsymbol{y}}$ in the convenience of computation. We also assume that for any $\boldsymbol{y} \in \Omega$, $\mathcal{U} = \operatorname{span}\{u_1(\cdot - \boldsymbol{y}), \ldots, u_m(\cdot - \boldsymbol{y})\}$. For example, the space of polynomials with degree no more than m would satisfy this assumption. We could then write the best approximation function $T_{\boldsymbol{y}}$ in the following form

$$T_{\boldsymbol{y}}(\boldsymbol{x}) = \sum_{j=1}^{m} c_j(\boldsymbol{y}) u_j(\boldsymbol{x} - \boldsymbol{y}), \quad \boldsymbol{x} \in \Omega,$$

where the coefficients $\boldsymbol{c}(\boldsymbol{y}) = (c_1(\boldsymbol{y}), c_2(\boldsymbol{y}), \ldots, c_m(\boldsymbol{y}))$ are given by

$$\boldsymbol{c}(\boldsymbol{y}) := \operatorname{argmin}_{\boldsymbol{a} \in \mathbb{R}^m} \sum_{i=1}^{N} \left[f_i - \sum_{j=1}^{m} a_j u_j(\boldsymbol{x}_i - \boldsymbol{y}) \right]^2 w(\|\boldsymbol{x}_i - \boldsymbol{y}\|). \tag{2.2}$$

We remark that it is a quadratic form and we are able to find the closed form of $\boldsymbol{c}(\boldsymbol{y})$. Let

$$\mathsf{G}(\boldsymbol{y}) := \left[\langle u_i(\cdot - \boldsymbol{y}), u_j(\cdot - \boldsymbol{y}) \rangle_{w_{\boldsymbol{y}}} \right]_{i,j=1}^{m},$$

and

$$\boldsymbol{L}_{\boldsymbol{y}}(f) = \left[\langle f, u_j(\cdot - \boldsymbol{y}) \rangle_{w_{\boldsymbol{y}}} \right]_{j=1}^{m}. \tag{2.3}$$

It follows from a direct calculation (also presented in [11,13]) that

$$\boldsymbol{c}(\boldsymbol{y}) = [\mathsf{G}(\boldsymbol{y})]^{-1} \boldsymbol{L}_{\boldsymbol{y}}(f). \tag{2.4}$$