Superconvergence of the discontinuous Galerkin method for nonlinear second-order initial-value problems for ordinary differential equations

Mahboub Baccouch [1]

Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182, United States

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A B S T R A C T

In this paper, we propose and analyze a superconvergent discontinuous Galerkin (DG) method for nonlinear second-order initial-value problems for ordinary differential equations. Optimal *a priori* error estimates for the solution and for the auxiliary variable that approximates the first-order derivative are derived in the L^2-norm. The order of convergence is proved to be $p + 1$, when piecewise polynomials of degree at most p are used. We further prove that the p-degree DG solutions are $\mathcal{O}(h^{2p+1})$ superconvergent at the downwind points. Finally, we prove that the DG solutions are superconvergent with order $p + 2$ to a particular projection of the exact solutions. The proofs are valid for arbitrary nonuniform regular meshes and for piecewise P^p polynomials with arbitrary $p \geq 1$. Computational results indicate that the theoretical orders of convergence and superconvergence are optimal.

1. Introduction

In this paper, we analyze a superconvergent discontinuous Galerkin (DG) method for the following general nonlinear second-order initial-value problem (IVP)

$$u'' = f(t, u, u'), \quad 0 \leq t \leq T, \quad u(0) = \alpha, \quad u'(0) = \beta, \tag{1.1}$$

where $u : [0, T] \to \mathbb{R}$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and α, β are given initial values. We assume that f sufficiently differentiable and satisfies a Lipschitz condition on the set $D = [0, T] \times \mathbb{R} \times \mathbb{R}$. The smoothness conditions for the function f depend on the degree of the polynomial. The precise conditions are described in later sections.

Many mathematical models in engineering and science applications are described by ordinary differential equations (ODEs). Nonlinear second-order IVPs of the form (1.1) are widely used in many disciplines. They are used to describe how physical systems change in time or space. Perhaps the most source of nonlinear second-order IVPs is Newton's law of motion, which governs the motion of everyday objects, with u'' representing acceleration and $f(t, u, u')$ force. For example, consider an object with constant mass m which is displaced a distance $u(t)$ along a one-dimensional path, when acted upon by a force F. Then Newton's second law of motion gives $mu'' = F(t, u, u')$, where F is the total force acting on the object, which may depend on the time t, the displacement u, and the velocity u'. Thus, the ODE governing its dynamics has the form (1.1) with $f(t, u, u') = \frac{1}{m} F(t, u, u')$. Many important dynamical systems are governed by an equation of motion that

E-mail address: mbaccouch@unomaha.edu.
[1] Fax: +1 402 554 2975.

is a nonlinear second-order ODE of the form (1.1). In particular, the damped oscillations of a rigid pendulum that rotates on a pivot subject to a uniform gravitational force in the vertical direction is governed by the nonlinear second-order ODE $u'' + cu' + b\sin(u) = 0$, where the unknown function $u(t)$ measures the angle between the pendulum and the vertical axis (angular displacement), $c \geq 0$ is the coefficient of friction, and the constant $b = \frac{g}{\ell}$, where g is the acceleration due to gravity and ℓ is the length of the pendulum. When the friction coefficient $c = 0$, the ODE describes the undamped oscillations of the pendulum.

Analytical solutions of IVPs of the form (1.1) exist only for simple dynamical systems, and therefore numerical methods are often required. There are many numerical schemes proposed in the literature for solving higher-order IVPs. Typically, IVPs for higher-order ODEs can be reformulated as a first-order system of equations. A wide range of numerical methods have been proposed in the literature for solving first-order systems. The most popular methods are Taylor and Runge–Kutta methods. Taylor schemes are obtained by truncating Taylor expansions. The practical difficulty of employing Taylor approximations is that they require to determine many derivatives. Runge–Kutta methods, which avoid the use of derivatives, are perhaps the most popular of the numerical methods for first-order systems because of their simplicity, relatively high accuracy, and broad applicability. Despite the popularity of high-order Runge–Kutta methods for integrating first-order systems of equations, they suffer from severe stability-based time step restrictions for very stiff problems. Here, we propose a DG method for solving (1.1), which has considerable advantages over the classical numerical method. The main advantages are: (i) it is A-stable, (ii) it is locally conservative, (iii) it exhibits strong superconvergence that can be used to estimate the discretization error, and (iv) it is suitable for hp-adaptivity.

DG method combines the best proprieties of the classical continuous finite element and finite volume methods such as consistency, flexibility, stability, conservation of local physical quantities, robustness and compactness. Recently, DG methods become highly attractive and popular, mainly because these methods are high-order accurate, nonlinear stable, highly parallelizable, easy to handle complicated geometries and boundary conditions, and capable to capture discontinuities without spurious oscillations. The original DG finite element method (FEM) was introduced in 1973 by Reed and Hill [33] for solving steady-state first-order linear hyperbolic problems. It provides an effective means of solving hyperbolic problems on unstructured meshes in a parallel computing environment. The discontinuous basis can capture shock waves and other discontinuities with accuracy [14,34]. The DG method can easily handle adaptivity strategies since the h-refinement (mesh refinement and coarsening) and the p-refinement (method order variation) can be done without taking into account the continuity restrictions typical of conforming FEMs. Moreover, the degree of the approximating polynomial can be easily changed from one element to the other [34]. Adaptivity is of particular importance in nonlinear hyperbolic problems given the complexity of the structure of the discontinuities and geometries involved. Due to local structure of DG methods, physical quantities such as mass, momentum, and energy are conserved locally through DG schemes. This property is very important for flow and transport problems. Furthermore, the DG method is highly parallelizable [27,26]. Because of these nice features, the DG method has been analyzed and extended to a wide range of applications. In particular, DG methods have been used to solve ODEs [7,23,28,30], hyperbolic [4,8,7,19,32,24,26,2,3,12,5,9] and diffusion and convection–diffusion [20,15–17] PDEs, to mention a few.

Related theoretical results in the literature including superconvergence results and error estimates of the DG methods for ODEs are given in [30,23,22,28,25,7,6,11]. In 1974, LaSaint and Raviart [30] presented the first error analysis of the DG method for first-order IVPs. They showed that the DG method is equivalent to an implicit Runge–Kutta method and proved a rate of convergence of $\mathcal{O}(h^p)$ for general triangulations and of $\mathcal{O}(h^{p+1})$ for Cartesian grids. Delfour et al. [23] investigated a class of Galerkin methods which lead to a family of one-step schemes generating approximations up to order $2p + 2$ for the solution of an ODE, when polynomials of degree p are used. In their proposed method, the numerical solution u_h at the discontinuity point t_n is defined as an average across the jump i.e., $\alpha_n u_h(t_n^-) + (1 - \alpha_n)u_h(t_n^+)$. By choosing special values of α_n, one can obtain the original DG scheme of LeSaint and Raviart [30] and Euler's explicit, improved, and implicit schemes. Delfour and Dubeau [22] introduced a family of discontinuous piecewise polynomial approximation schemes. They presented a more general framework of one-step methods such as implicit Runge–Kutta and Crank–Nicholson schemes, multi-step methods such as Adams–Bashforth and Adams–Moulton schemes, and hybrid methods. Later, Johnson [28] proved new optimal *a priori* error estimates for a class of implicit one-step methods for stiff ODEs obtained by using the DG method with piecewise polynomials of degree zero and one. Johnson and Pitkäranta [29] proved a rate of convergence of $\mathcal{O}(h^{p+1/2})$ for general triangulations and Peterson [32] confirmed this rate to be optimal. Richter [35] obtained the optimal rate of convergence $\mathcal{O}(h^{p+1})$ for some structured two-dimensional non-Cartesian grids. We also would like to mention the work of Estep [25], where the author outlined a rigorous theory of global error control for the approximation of first-order IVP. In [7], Adjerid et al. showed that the DG solution of one-dimensional hyperbolic problems exhibit an $\mathcal{O}(h^{p+2})$ superconvergence rate at the roots of the right Radau polynomial of degree $p + 1$. Furthermore, they obtained a $(2p + 1)$-th order superconvergence rate of the DG approximation at the downwind point of each element. They performed a local error analysis and showed that the local error on each element is proportional to a Radau polynomial. They further constructed implicit residual-based *a posteriori* error estimates but they did not prove their asymptotic exactness. Adjerid and Temimi [6], proposed a DG method to solve higher-order IVPs for linear ODEs. They proved that the DG exhibits an optimal $\mathcal{O}(h^{p+1})$ convergence rate in the L^2-norm. They further showed that the p-degree DG solution of differential equation of order m and its first $m - 1$ derivatives are $\mathcal{O}(h^{2p+2-m})$ superconvergent at the end of each step. They also established that the p-degree DG solution is $\mathcal{O}(h^{p+2})$ superconvergent at the roots of $(p + 1 - m)$-degree Jacobi polynomial on each step. More recently, the author [11] constructed and analyzed *a posteriori* error estimate of the discretization errors for the DG