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Discontinuous Galerkin time stepping method for solving linear space fractional partial differential equations

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ABSTRACT

In this paper, we consider the discontinuous Galerkin time stepping method for solving the linear space fractional partial differential equations. The space fractional derivatives are defined by using Riesz fractional derivative. The space variable is discretized by means of a Galerkin finite element method and the time variable is discretized by the discontinuous Galerkin method. The approximate solution will be sought as a piecewise polynomial function in t of degree at most $q-1, q \geq 1$, which is not necessarily continuous at the nodes of the defining partition. The error estimates in the fully discrete case are obtained and the numerical examples are given.

1. Introduction

In this paper we will consider the discontinuous Galerkin time stepping methods for solving the following linear space fractional partial differential equation, with $1/2 < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^{2\alpha} u(t,x)}{\partial |x|^{2\alpha}} = f(t,x), \quad 0 < t < T, \; 0 < x < 1, \tag{1}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{2}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1, \tag{3}$$

where the Riesz fractional derivative is defined by, [31,33]

$$\frac{\partial^{2\alpha} w(x)}{\partial |x|^{2\alpha}} = -\frac{1}{2\cos(\alpha\pi)} \Big({}_0^R D_x^{2\alpha} w(x) + {}_x^R D_1^{2\alpha} w(x) \Big),$$

and ${}_0^R D_x^{\gamma} w(x)$ and ${}_x^R D_1^{\gamma} w(x), 1 < \gamma < 2$ are called the left-sided and right-sided Riemann–Liouville fractional derivatives, respectively,

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$$ {}_0^R D_x^{\gamma} w(x) = \frac{1}{\Gamma(2-\gamma)} \frac{d^2}{dx^2} \int_0^x (x-y)^{1-\gamma} w(y)\, dy, \quad {}_x^R D_1^{\gamma} w(x) = \frac{1}{\Gamma(2-\gamma)} \frac{d^2}{dx^2} \int_x^1 (y-x)^{1-\gamma} w(y)\, dy. \tag{4} $$

Space fractional partial differential equations are widely used to model complex phenomena, for example, in quasi-geostrophic flow, the fast rotating fluids, the dynamic of the frontogenesis in meteorology, the diffusions in fractal or disordered medium, the pollution problems, the mathematical finance and the transport problems, soil contamination and underground water flow, see, e.g., [5,9,24,4,29].

In recent years, many authors consider the numerical methods for solving space fractional partial differential equations, e.g., finite difference methods [1,2,27,28,34–38,40,19,30], finite element methods [11–18,32,42,41] and spectral methods [25, 26,7,8], matrix transfer technique (MTT) [20,21]. Recently, Jin et al. [23] considered the finite element method for solving the linear space fractional parabolic equation where the space fractional derivative is defined as left-sided Riemann–Liouville derivative, see also [22]. The estimates in [23] are for both smooth and nonsmooth initial data, and are expressed directly in terms of the smoothness of the initial data.

The Riesz space fractional partial differential equations were firstly proposed by Chaves [10] to investigate the mechanism of super-diffusion. Benson et al. [3,4] considered the fractional order governing equation of Lévy motion. Zhang et al. [42] considered a finite element method in space and backward difference method in time for solving Riesz space fractional partial differential equation. Sousa [36] studied a second order numerical method for Riesz space fractional convection–diffusion equation. Bu et al. [6] considered a finite element method in space and Crank–Nicolson method in time for solving Riesz space fractional partial differential equations in two-dimensional case. Duan et al. [13] studied a finite element method in space and backward Euler method in time for solving Riesz space fractional partial differential equations in two-dimensional case.

In this paper, we will consider a finite element method in space and a discontinuous Galerkin method in time for solving Riesz space fractional partial differential equation. When the approximating functions are piecewise constant in time, we proved that the error is $O(h^{r-\alpha} + k_n)$ and the bounds contain the terms $\|u\|_{r,J_n}$ and $\|u_t\|_{\alpha,J_n}$, see Theorem 4.1 below. When the approximating functions are piecewise linear in time, we proved that the error is $O(h^{2(r-\alpha)} + k_n^3)$ and the bounds contain the terms $\|u\|_{r,J_n}$ and $\|u_{tt}\|_{r,J_n}$, see Theorem 4.3 below. The advantages of the discontinuous Galerkin method is that, e.g., variable coefficients and nonlinearities present no complication in principle. We obtain precise error estimates for the discontinuous Galerkin method which make it possible to construct the adaptive methods based on the automatic time-step control.

The paper is organized as follows. In Section 2, we introduce some fractional Sobolev spaces and some basic lemmas. In Section 3, we give the error estimates for the backward Euler method. In Section 4, we consider the error estimates for the discontinuous Galerkin time stepping method for $q = 1, 2$. Finally in Section 4, we give two numerical examples.

By C we denote a positive constant independent of the functions and parameters concerned, but not necessarily the same at different occurrences.

2. Preliminaries

In this section, we will introduce some fractional Sobolev spaces.

Definition 2.1. [14,25] For any $\sigma > 0$, we define the spaces ${}^l H_0^{\sigma}(0,1)$ and ${}^r H_0^{\sigma}(0,1)$ to be the closures of $C_0^{\infty}(0,1)$ with respect to the norms $\|v\|_{{}^l H_0^{\sigma}(0,1)}$ and $\|v\|_{{}^r H_0^{\sigma}(0,1)}$, respectively, where

$$ \|v\|^2_{{}^l H_0^{\sigma}(0,1)} := \|v\|^2_{L^2(0,1)} + \|{}_0^R D_x^{\sigma} v\|^2_{L^2(0,1)}, $$

and

$$ \|v\|^2_{{}^r H_0^{\sigma}(0,1)} := \|v\|^2_{L^2(0,1)} + \|{}_x^R D_1^{\sigma} v\|^2_{L^2(0,1)}. $$

The semi-norms are defined by $|v|_{{}^l H_0^{\sigma}(0,1)} := \|{}_0^R D_x^{\sigma} v\|_{L^2(0,1)}$ and $|v|_{{}^r H_0^{\sigma}(0,1)} := \|{}_x^R D_1^{\sigma} v\|_{L^2(0,1)}$, respectively.

Remark 2.1. In Definition 2.1, $|v|_{{}^l H_0^{\sigma}(0,1)}, \sigma > 0$ is a semi-norm (not a norm) since $|v|_{{}^l H_0^{\sigma}(0,1)} = 0$ does not imply $v = 0$. For example, when $0 < \sigma < 1$, let $w(x) = x^{\sigma-1}$, we have $w(x) \neq 0$ and

$$ {}_0^R D_x^{\sigma} w(x) = \frac{1}{\Gamma(1-\sigma)} \frac{d}{dx} \int_0^x (x-y)^{-\sigma} w(y)\, dy = \frac{1}{\Gamma(1-\sigma)} \frac{d}{dx} \int_0^x (x-y)^{-\sigma} y^{\sigma-1}\, dy $$

$$ = \frac{1}{\Gamma(1-\sigma)} \frac{d}{dx} \int_0^1 t^{-\sigma} (1-t)^{\sigma-1}\, dt = 0, $$