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Block boundary value methods applied to functional differential equations with piecewise continuous arguments ☆

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ABSTRACT

This paper deals with a class of functional differential equations with piecewise continuous arguments. Block boundary value methods (BBVMs) are extended to solve this class of equations. It is shown under the Lipschitz condition that the order of convergence of an extended block boundary value method coincides with its order of consistency. Moreover, we study the linear stability of the extended methods and give the corresponding asymptotical stability criterion. In the end, with several numerical examples, the theoretical results and the computational effectiveness of the methods are further illustrated.

1. Introduction

This paper will be concerned with initial value problems for functional differential equations with piecewise continuous arguments (FDEPCAs):

$$y'(t) = f\left(t, y(t), y(\lfloor t \rfloor)\right), \quad t \in [0, +\infty); \qquad y(0) = y_0, \tag{1.1}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function, $y,\ y_0 \in \mathbb{R}^d$, and it is assumed that function $f : [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is smooth enough and satisfies the Lipschitz condition for all $t \in [0, +\infty)$, $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$:

$$\| f(t, y_1, z_1) - f(t, y_2, z_2)\|_\infty \le L_1 \|y_1 - y_2\|_\infty + L_2 \|z_1 - z_2\|_\infty. \tag{1.2}$$

The solution of (1.1) can be defined as follows.

Definition 1.1. (Cf. [24]) A solution of (1.1) on $[0, +\infty)$ is a function $y(t)$ that satisfies the following conditions:

(i) $y(t)$ is continuous on $[0, +\infty)$;
(ii) the derivative $y'(t)$ exists at each point $t \in [0, +\infty)$, with the possible exception of the points $\lfloor t \rfloor \in [0, +\infty)$ where one-sided derivatives exist;
(iii) equation (1.1) is satisfied on each interval $[n, n+1)$ $(n = 0, 1, 2, \ldots)$.

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Problems (1.1) are useful to describe certain actual phenomena from physics, biology, control science and the other scientific fields. Some analytical results for this kind of problems have been presented by many authors (see e.g. [10–12]). A detailed introduction to the theory of FDEPCAs can be found in Wiener's monograph [24].

Generally speaking, however, it is difficult or even impossible to obtain an exact solution of problem (1.1). Hence, in the recent years, efforts have been made to develop various numerical methods to solve this kind of problems and study the corresponding algorithmic theory. In [15,17–19], Liu, Song et al. constructed Runge–Kutta methods to solve linear FDEPCAs and derived a number of stability and oscillation results. Wang [21,22] and Liu [16] further extended Runge–Kutta methods and their theory to nonlinear situation of FDEPCAs. Moreover, Wen & Li [23] and Song & Liu [20] considered using linear multistep methods to solve problem (1.1) and analyzed the linear and nonlinear stability.

The above approach focused on Runge–Kutta methods and linear multistep methods. They did not involve boundary value methods (BVMs). In fact, BVMs and the induced BBVMs have been verified to be very effective numerical methods for solving various initial or boundary value problems of differential equations (see e.g. [1–6,13,14]), whose elementary theory refers to Brugnano & Trigiante's monograph [7]. Furthermore, Zhang, Chen et al. adapted BVMs and BBVMs to deal with delay differential equations (cf. [27]), delay differential-algebraic equations (cf. [28,29]), Volterra integral and integro-differential equations (cf. [8]) and delay Volterra integro-differential equations (cf. [9]). Subsequently, Xu, Zhao and Gao investigated stability of BBVMs for neutral pantograph equations and neutral multi-delay differential equations in [25,26], respectively.

Although BVMs and BBVMs have been applied to function differential equations with delay, only the cases of the equations with constant delay or proportional delay were concerned. As we know, up to now, no result has been presented for BBVMs applied to problems (1.1). Hence, in the present paper, we will adapt BBVMs for problems (1.1). The outline of this paper is as follow. In Section 2, we will give a brief review to BBVMs and then extend this kind of methods to solve problems (1.1). In Section 3, we will analyze the convergence of the extended BBVMs and derive a convergence criterion. In Section 4, we will study the asymptotical stability of the methods and give a sufficient condition for the numerical stability. In Section 5, with several numerical examples, we will further illustrate the theoretical results and computational effectiveness of the methods.

2. BBVMs applied to FDEPCAs

In this section, we will consider adapting the underlying BVMs to solve FDEPCAs (1.1) on $[0, T]$. For convenience, we first give a brief review to the underlying BVMs and the corresponding BBVMs for the d-dimensional problems of ODEs

$$y'(t) = f(t, y), \quad t \in [0, T]; \quad y(0) = y_0. \tag{2.1}$$

A full introduction to these methods can be found in Brugnano and Trigiante's monograph [7].

Let $0 = t_0 < t_1 < \cdots < t_s = T$ be a uniform mesh on $[0, T]$ with $t_i = t_0 + ih$, $i = 0, 1, \ldots, s$ and $h = T/s$. Then, the problem (2.1) can be approximated by a k-step BVM with k_1 initial conditions and k_2 $(= k - k_1)$ final conditions, that is

$$\sum_{j=-i}^{k-i} \alpha_{i+j}^{(i)} \; y_{i+j} \;=\; h \sum_{j=-i}^{k-i} \beta_{i+j}^{(i)} \; f_{i+j}, \qquad i = 1, \cdots, k_1 - 1, \tag{2.2}$$

$$\sum_{j=-k_1}^{k_2} \alpha_{i+j} \; y_{i+j} \;=\; h \sum_{j=-k_1}^{k_2} \beta_{i+j} \; f_{i+j}, \qquad i = k_1, \cdots, s - k_2, \tag{2.3}$$

$$\sum_{j=-i}^{k-i} \alpha_{i+j}^{(i)} y_{s-k+i+j} = h \sum_{j=-i}^{k-i} \beta_{i+j}^{(i)} f_{s-k+i+j}, \quad i = s - k_2 + 1, \cdots, s, \tag{2.4}$$

where α_i, $\alpha_{i+j}^{(i)}$, β_i, $\beta_{i+j}^{(i)}$ are some given real coefficients, and y_i and f_i are approximations to $y(t_i)$ and $f(t_i, y(t_i))$, respectively. The scheme (2.3) is called *main scheme*, (2.2) and (2.4) are called *auxiliary schemes*, and they are assumed to have the same local order. By introducing the notations

$$\mathbf{y} = (y_1^T, y_2^T, \cdots, y_s^T)^T, \quad F(\mathbf{y}) = \Big(f(t_1, y_1)^T, f(t_2, y_2)^T, \cdots, f(t_s, y_s)^T \Big)^T,$$

method (2.2)–(2.4) can be rewritten in a compact form

$$(A \otimes I_d)\mathbf{y} - h(B \otimes I_d)F(\mathbf{y}) = -\mathbf{a_0} \otimes y_0 + h\mathbf{b_0} \otimes f(t_0, y_0), \tag{2.5}$$

where I_d is the $d \times d$ identity matrix, $\otimes$ denotes the Kronecker product, $A^e = [\mathbf{a_0}|A] \in R^{s\times(s+1)}$ is given by