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Extending the method of fundamental solutions to non-homogeneous elastic wave problems

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ABSTRACT

Two meshfree methods are developed for the numerical solution of the non-homogeneous Cauchy–Navier equations of elastodynamics in an isotropic material. The two approaches differ upon the choice of the basis functions used for the approximation of the unknown wave amplitude. In the first case, the solution is approximated in terms of a linear combination of fundamental solutions of the Navier differential operator with different source points and test frequencies. In the second method the solution is approximated by superposition of acoustic waves, i.e. fundamental solutions of the Helmholtz operator, with different source points and test frequencies. The applicability of the two methods is justified in terms of density results and a convergence result is proven. The accuracy of the methods is illustrated through 2D numerical examples. Applications to interior elastic wave scattering problems are also presented.

1. Introduction

The method of fundamental solutions (MFS) is a meshfree numerical algorithm for the approximate solution of elliptic Boundary Value Problems (BVP) involving a homogeneous partial differential equation (PDE) with a known fundamental solution. In the classical approach, the solution of the BVP is approximated by superposition of fundamental solutions with source points (singularities) located in the exterior of the domain of interest. This idea was first introduced by Kupradze and Aleksidze [25] and later formulated as a numerical technique by Mathon and Johnston [29]. Due to its simple formulation and high precision the MFS has recently gained a lot of attention from the scientific and engineering community and has been applied to a variety of physical problems in fluid mechanics, linear elasticity, acoustics, electromagnetism, stationary heat conduction and option pricing. In particular, the MFS has been used for the numerical solution of direct [23,34,35] and inverse [27,26] boundary value problems in elastostatics and thermo-elastostatics. For a list of applications of the MFS and related variants we recommend the survey papers [10,11,15,22] and the book [14].

A boundary value problem involving a non-homogeneous PDE is usually solved in two steps. First a particular solution of the PDE is derived, e.g. using the Dual Reciprocity Method (DRM) [32], and then the associated homogeneous BVP is solved for the homogeneous part of the solution. Here, the MFS may be used in the second step [14] noting that the global

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accuracy of the total solution will depend on the quality of the approximate particular solution. Most commonly, Radial Basis Functions (RBF) are used as basis functions in the Dual Reciprocity Method, e.g. thin plate splines [13], polyharmonic splines [31,38] or multiquadrics [9,16]. However, besides the typical ill-conditioning problem associated with RBF interpolation, obtaining a closed analytic form for the particular solution may be a time consuming and mathematically elaborate process.

An alternative two step method using only fundamental solutions as basis functions was proposed in [2] for Poisson and non-homogeneous Helmholtz problems. In this publication the source term was approximated by a linear combination of Helmholtz fundamental solutions with different test frequencies and source points. Density results in the L^2 space were proven in order to justify the approach from a theoretical point of view. The corresponding particular solution was then obtained, at no extra computational cost, by re-scaling the coefficients of the linear combination. Finally the homogeneous solution was calculated by the classical MFS. An efficient form of this method, based on a matrix decomposition algorithm, was presented in [21] for polyharmonic problems in axisymmetric domains. The numerical procedure from [2] was adopted in [5] for the solution of non-homogeneous Helmholtz problems, using plane acoustic waves with different frequencies and directions of propagation. In [28] plane elastic waves were used in order to approximate the particular solution for a non-homogeneous Brinkman flow problem.

A unified (one stage) meshfree method, based on the solution of a single linear system, where the PDE and the boundary conditions are collocated simultaneously using RBFs was introduced by Kansa [18,19]. In [8] the method of approximate particular solutions was developed, where the collocation was performed using two sets of RBFs, related in the sense of the DRM. In [36,37] a linear combination of RBFs and fundamental solutions was used for the solution of nonlinear Poisson problems. A set of Helmholtz fundamental solutions with different source points and test frequencies was used in the MFS-K method [6] for the non-homogeneous acoustic wave propagation problem. Such basis functions are particularly appropriate for the collocation of Helmholtz type PDEs since they simplify the evaluation of the Laplace operator and differentiation is avoided. An efficient algorithm for solving the MFS-K collocation linear system for BVPs in axisymmetric domains was presented in [20].

Another advantage of the MFS-K method is that it can be used for the solution of a fairly general class of Helmholtz type PDEs with non-constant coefficients. For instance, acoustic wave problems posed in domains with variable speeds of wave propagation can be solved. As a consequence, the method has also been successfully coupled with time discretization schemes and applied for the solution of heat conduction problems for materials with time and space dependent properties [39] and time-fractional inverse diffusion problems [41,42].

Here we propose two meshfree methods which may be viewed as extensions of the MFS-K method from the scalar acoustic to the vector elastic wave propagation problem. In particular, these methods will be applied for the approximate solution of BVPs for the non-homogeneous Cauchy–Navier equations of elastodynamics, posed in bounded isotropic materials.

In the first method, later on referred to as MFS-K, the unknown solution of the BVP will be approximated by a linear combination of Kupradze tensors (fundamental solutions of the Navier operator) with different source points and test frequencies. Appropriate density results will also be proved in order to justify the convergence of the method. This method is the natural extension of the method presented in [6].

In the second variant of the MFS that is developed here, a termwise approximation of the unknown displacement field is considered. In particular, each component of the unknown (vector) solution is approximated by superposition of acoustic waves, i.e. fundamental solutions of the Helmholtz operator. We will refer to this method as the MFS-H. In view of its formulation, this method is similar to Kansa's RBF methods. Nevertheless, the use of Helmholtz fundamental solutions is more appropriate than the use of general purpose RBFs since elastic waves can be split into P-waves and S-waves, each one satisfying a Helmholtz equation.

In Section 2 we recall the classical MFS applied to homogeneous elastic wave problems. In Section 3 the formulations of the proposed methods for the non-homogeneous problem are presented. In Section 4 the density results required in order to justify the convergence of the MFS-K are proven. Finally, in Section 5, we present three numerical examples illustrating the convergence of the two methods.

2. Elastic wave scattering in isotropic media

Consider a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a class C^1 boundary $\Gamma = \partial\Omega$. We denote the linear elastic (Lamé) operator by $\mathcal{A} = \nabla \cdot \sigma$, where

$$\sigma(u) = (\lambda \nabla \cdot u)I + \mu(\nabla u + \nabla u^\top) \tag{1}$$

is the stress tensor and $\lambda > 0$ and $\mu > 0$ are the Lamé parameters.

For a given source function f, the time harmonic elastic wave with constant frequency $\omega > 0$ satisfies the inhomogeneous Navier equation, see [24],

$$\mathcal{A}u + \rho\omega^2 u = \mu\Delta u + (\lambda + \mu)\nabla\nabla \cdot u + \rho\omega^2 u = f, \tag{2}$$

where $\rho > 0$ denotes the variable density of the elastic material.

The scattered wave u in Ω, induced by a boundary function g that depends on the incident elastic wave input, is then calculated by solving the BVP