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High order exponentially fitted methods for Volterra integral equations with periodic solution

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ABSTRACT

The present paper illustrates the construction of direct quadrature methods of arbitrary order for Volterra integral equations with periodic solution. The coefficients of these methods depend on the parameters of the problem, following the exponential fitting theory. The convergence of these methods is analyzed, and some numerical experiments are illustrated to confirm theoretical expectations and for comparison with other existing methods.

1. Introduction

In this paper we consider Volterra integral equations (VIEs) with periodic solution of the type

$$\begin{aligned} y(x) &= f(x) + \int_{-\infty}^{x} k(x-s)y(s)ds, \qquad x \in [0, x_{end}] \\ y(x) &= \psi(x), \qquad -\infty < x \le 0, \end{aligned} \tag{1.1}$$

where $k \in L^1(\mathbb{R}^+)$, f is continuous and T-periodic on $[0, x_{end}]$, ψ is continuous and bounded on $\mathbb{R}^-$. Under suitable hypotheses on the kernel k, (1.1) has a unique T-periodic solution [4]. VIEs of type (1.1) model periodic physical and biological processes with memory, like for example seasonal biological phenomena [1,19] and the response of a nonlinear circuit to a periodic input [21]. Further examples are furnished in [4,8,10].

An efficient and accurate numerical solution of (1.1) may be found by means of special purpose methods, which exploit the *a priori* knowledge of the qualitative behavior of the solution. On this direction, two main numerical schemes have been proposed. In [2,4] a collocation method based on a mixed interpolation technique is illustrated. In [9,10,12] direct quadrature (DQ) methods based on the exponential fitting technique [16,18] were introduced and analyzed. Exponential fitting is a powerful technique which allows to derive accurate and efficient numerical schemes also for high oscillatory problems, in the context of ordinary differential equations (ODEs), partial differential equations, quadrature, interpolation [7,14–18]. A recent review on the exponential fitting theory is [20]. In general, exponentially fitted (ef) methods have coefficients depending on some estimates of parameters of the problem itself (like the frequency of periodic solutions), and tend to the classical numerical methods when the problem is not oscillatory, thus they may be considered as a generalization of classical numerical schemes.

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The aim of the paper is the construction and analysis of a family of DQ methods based on exponentially-fitted Gaussian quadrature formulae. Recently, an ef-DQ method based on a two-nodes Gaussian-type formula has been introduced in [10] (see also [12]). Here we go one step further, and propose a systematic approach for the construction of ef-DQ Gaussian methods of arbitrary order, to achieve a better accuracy. First, we introduce a general class of ef Gaussian quadrature rules, based on an exponential fitting space. Since a direct quadrature method based on such formulas requires an approximation of the solution at points not belonging to the mesh, we introduce an interpolation approximation based on a suitable trigonometric basis. The final method may achieve an high order of convergence, which depends on the number of nodes of the quadrature rule and on the interpolation approximation accuracy.

The paper is organized as follows. In Sec. 2 we introduce and analyze the convergence of a family of ef-DQ methods based on Gaussian-type rules and on a suitable interpolation technique. In Sec. 3 we illustrate the construction of ef-DQ methods of order six. The performances of the methods are illustrated in 4. Last section contains some concluding remarks.

2. Exponentially fitted Gaussian quadrature rule

For sake of completeness, we report the main results which guarantee the existence and uniqueness of solution of a periodic solution for the problem (1.1) [4]. In the following we will assume that hypotheses of Theorem 2.2 are satisfied.

Theorem 2.1. *Consider the periodically forced VIE:*

$$y(x) = f(x) + \int_{-\infty}^{x} k(x-s)y(s)ds, \qquad x \in \mathbb{R}, \tag{2.1}$$

where $k : \mathbb{R} \to \mathbb{R}$, $k \in L^1(0,\infty)$ and $k(x) = 0$ for $x < 0$, f is continuous and T-periodic. Let $H(s) = \det(I - \hat{K}(s))$, where $\hat{K}(s)$ is the Laplace transform of k. If $H(s)$ does not vanish on the imaginary axis, then (2.1) has a unique continuous T-periodic solution $\phi(x)$.

Theorem 2.2. *Assume that hypotheses of Theorem 2.1 hold. Then the solution of the initial value problem (1.1) is periodic if and only if the initial function $\psi(x)$ is the unique solution $\phi(x)$ of equation (2.1).*

With the purpose of well reproducing the problem modeled by (1.1), we approximate the whole integrand of (1.1) by suitable exponential and trigonometric functions. To this purpose we consider as *test equation* the problem (1.1) with kernel

$$k(x) = e^{\alpha x}, \tag{2.2}$$

and $f(x)$ such that

$$y(x) = \sum_{k=0}^{P-1} \left(c_{1,k} x^k \cos(\omega x) + c_{2,k} x^k \sin(\omega x) \right), \tag{2.3}$$

where $P \geq 1$, $\omega, c_{1,k}, c_{2,k} \in \mathbb{R}$, $\forall k$. It is possible to verify that, if $f(x)$ and $\psi(x)$ have the same structure as $y(x)$, and $k(x)$ is of the type (2.2), then the solution of (1.1) has the form (2.3). In the spirit of exponential fitting theory, we derive a DQ method which is exact when applied to this test equation, and will be proved to be more accurate than general-purpose methods when applied to more general periodic or oscillatory problems.

The first step to construct such DQ method is to formulate an adapted quadrature rule for the integral in (1.1), when the kernel and the solution are given by (2.2) and (2.3), respectively. We construct a Gauss quadrature formula for the integral

$$I[g](X) = \int_{X-h}^{X+h} g(x)dx,$$

where $X > 0$ and $h > 0$, which is exact on the fitting space

$$\mathcal{B} := \{x^k e^{(\alpha \pm i\omega)x}, \ k = 0, \ldots, P-1\}. \tag{2.4}$$

The quadrature formula is of type

$$Q[g](X) := h \sum_{k=0}^{P-1} a_k g(X + \xi_k h) \tag{2.5}$$

where the weights and nodes

$$a_k = a_k(\alpha h, \omega h), \quad \xi_k = \xi_k(\alpha h, \omega h), \qquad k = 0, 1, \ldots, P-1, \tag{2.6}$$