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On the simultaneous approximation of a Hilbert transform and its derivatives on the real semiaxis

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ABSTRACT

In this paper we propose a global method to approximate the derivatives of the weighted Hilbert transform of a given function f

$$\mathbf{H}_p(fw_\alpha,t)=\frac{d^p}{dt^p}\int_0^{+\infty}\frac{f(x)}{x-t}w_\alpha(x)dx=p!\,\oint_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx,$$

where $p\in\{1,2,\dots\}$, $t>0$, and $w_\alpha(x)=e^{-x}x^\alpha$ is a Laguerre weight. The right-hand integral is defined as the finite part in the Hadamard sense. The proposed numerical approach is convenient when the approximation of the function $\mathbf{H}_p(fw_\alpha,t)$ is required. Moreover, if there is the need, all the computations can be performed without differentiating the density function f. Numerical stability and convergence are proved in suitable weighted uniform spaces and numerical tests which confirm the theoretical estimates are presented.

1. Introduction

The paper is devoted to the approximation of the derivatives of the weighted Hilbert transform of f

$$\mathbf{H}_p(fw,t)=\frac{d^p}{dt^p}\int_0^{+\infty}\frac{f(x)}{x-t}w(x)dx=p!\,\oint_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}w(x)dx,\tag{1}$$

where $p\in\{1,2,\dots\}$, $t>0$, $w(x):=w_\alpha(x)=e^{-x}x^\alpha$ is a Laguerre weight. The integral in (1) can be also defined as a finite part integral in the Hadamard sense (see [7,16]). Integrals of the type (1) appear for instance in hypersingular integral equations, models for many problems in Physics and Engineering areas (see [16] and the reference therein, [5,10,1]). Usually, in the literature, quadrature rules are proposed for the approximation of $\mathbf{H}_p(fw,t)$ for any fixed t. Instead, in the present paper, setting

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$$\mathbf{H}_p(fw,t) = \frac{d^p}{dt^p}\left(\int_0^{+\infty} \frac{f(x)-f(t)}{x-t} w(x)dx + f(t) \,⨍_0^{+\infty} \frac{w(x)}{x-t}dx\right)$$
$$=: \frac{d^p}{dt^p}\mathbf{F}(f,t) + \frac{d^p}{dt^p}\left(f(t)\mathbf{H}_0(w,t)\right), \qquad (2)$$

we propose to approximate the function $\mathbf{F}^{(p)}(f)$ by the p-th derivative of a suitable Lagrange polynomial interpolating $\mathbf{F}(f)$ at Laguerre zeros. For a correct error estimate in weighted uniform spaces, at first we determine the class of $\mathbf{F}(f)$ depending on the Zygmund-type space f belongs to. Since in the general case the samples of $\mathbf{F}(f)$ at the interpolation knots cannot be exactly computed, we approximate them by a truncated Gauss–Laguerre rule (see [12]). Moreover, by reusing the same interpolation knots, it is possible approximate also the p-th derivative of the function $f(t)\mathbf{H}_0(w,t)$, avoiding the differentiation of the density function f.

This procedure is especially advisable when the approximation of $\mathbf{H}_p(fw,t)$ is required for a “large” number of t and/or the uniform convergence of the rule to $H_p(fw)$ is needed. This happens, for instance, when (1) appears in a hypersingular integral equation and in order to solve it one wants to use a collocation method.

The paper is organized as follows. In Section 2 there are collected some auxiliary results and notations. Section 3 provides the exposition of the numerical methods and results about the stability and the convergence, with error estimates in some weighted uniform spaces. Section 4 contains a brief description of computational details in the implementation process. In Section 5 some numerical experiments are discussed and comparisons with some standard numerical methods are shown. Finally in Section 6 the proofs of our main results are stated.

2. Basic results and properties

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a,b,\ldots$, and $\mathcal{C} = \mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ depends on $a,b,\ldots$. Moreover, if $A,B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $0 < \mathcal{C} \neq \mathcal{C}(A,B)$ such that $\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B$. Finally, $\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m.

Let $w(x) = e^{-x}x^{\alpha}$ be the Laguerre weight of parameter $\alpha > -1$ and let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. Let us denote by $\{x_{m,k}\}_{k=1}^m$ the zeros of $p_m(w)$ in increasing order, i.e. $x_{m,k} < x_{m,k+1}, k = 1,\ldots,m-1$. From now on, for any fixed $0 < \theta < 1$, the integer j will denote the index of the zero of $p_m(w)$ s.t.

$$j := j(m) = \min_{k=1,2,..,m} \left\{k : x_{m,k} \geq 4m\theta\right\}. \qquad (3)$$

With $u(x) = x^{\gamma} e^{-x/2}, \gamma \geq 0$, we will consider

$$C_u = \begin{cases} \{f \in C^0((0,\infty)) : \lim\limits_{\substack{x\to+\infty \\ x\to 0^+}} (fu)(x) = 0\}, & \gamma > 0, \\ \left\{f \in C^0([0,\infty)) : \lim\limits_{x\to+\infty} (fu)(x) = 0\right\}, & \gamma = 0, \end{cases}$$

equipped with the norm

$$\|f\|_{C_u} = \|fu\| := \|fu\|_\infty = \sup_{x\geq 0} |(fu)(x)|,$$

where $C^0(E)$ is the space of the continuous functions on the set E. Sometimes, for the sake of brevity, we will use $\|f\|_E = \sup_{x\in E} |f(x)|$.

For smoother functions, we introduce the Sobolev-type spaces of order $r \in \mathbb{N}$

$$W_r(u) = \left\{f \in C_u \ : \ f^{(r-1)} \in AC(0,+\infty) \text{ and } \|f^{(r)}\varphi^r u\| < +\infty\right\},$$

where $\varphi(x) = \sqrt{x}$ and $AC((0,+\infty))$ is the set of the absolutely continuous functions on every closed subset of $(0,+\infty)$. We equip them with the norm

$$\|f\|_{W_r(u)} := \|fu\| + \|f^{(r)}\varphi^r u\|.$$

In what follows $W_0(u) = C_u$. For any $f \in C_u$ and for any $t > 0$, let

$$\Omega^r_\varphi(f,t)_u = \sup_{0<h\leq t} \|u\Delta^r_{h\varphi} f\|_{I_{rh}}$$