Applied Numerical Mathematics 113 (2017) 1–25

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

High-order, stable, and efficient pseudospectral method using barycentric Gegenbauer quadratures

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ARTICLE INFO

Article history:
Received 23 December 2015
Received in revised form 31 August 2016
Accepted 25 October 2016
Available online 29 October 2016

Keywords:
Barycentric interpolation
Gegenbauer polynomials
Gegenbauer quadrature
Integration matrix
Pseudospectral method

ABSTRACT

The work reported in this article presents a high-order, stable, and efficient Gegenbauer pseudospectral method to solve numerically a wide variety of mathematical models. The proposed numerical scheme exploits the stability and the well-conditioning of the numerical integration operators to produce well-conditioned systems of algebraic equations, which can be solved easily using standard algebraic system solvers. The core of the work lies in the derivation of novel and stable Gegenbauer quadratures based on the stable barycentric representation of Lagrange interpolating polynomials and the explicit barycentric weights for the Gegenbauer–Gauss (GG) points. A rigorous error and convergence analysis of the proposed quadratures is presented along with a detailed set of pseudocodes for the established computational algorithms. The proposed numerical scheme leads to a reduction in the computational cost and time complexity required for computing the numerical quadrature while sharing the same exponential order of accuracy achieved by Elgindy and Smith-Miles [14]. The bulk of the work includes three numerical test examples to assess the efficiency and accuracy of the numerical scheme. The present method provides a strong addition to the arsenal of numerical pseudospectral methods, and can be extended to solve a wide range of problems arising in numerous applications.

1. Introduction

The past few decades have seen a conspicuous attention towards the solution of differential problems by working on their integral reformulations; cf. [6–9,15,16,18,21,26,28,31]. Perhaps one of the reasons that laid the foundation of this methodology appears in the well stability and boundedness of numerical integral operators in general whereas numerical differential operators are inherently ill-conditioned; cf. [10,19,21]. The numerical integral operator used in the popular pseudospectral methods is widely known as the spectral integration matrix (also called the operational matrix of integration), which dates back to El-Gendi [8] in the year 1969. In fact, the introduction of the numerical integration matrix has provided the key to apply the rich and powerful matrix linear algebra in many areas [10].

In 2013, Elgindy and Smith-Miles [14] presented some novel numerical quadratures based on the concept of numerical integration matrices. Their unified approach employed the Gegenbauer basis polynomials to achieve rapid convergence rates for small/medium range of spectral expansion terms while using Chebyshev and Legendre bases polynomials for a large-scale number of expansion terms. The established quadratures were presented in basis form, and were parameter optimized in the sense of minimizing the Gegenbauer parameter associated with the quadrature truncation error. This key idea allowed for interpolating the integrand function at some Gegenbauer–Gauss (GG) sets of points called the adjoint GG points instead

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http://dx.doi.org/10.1016/j.apnum.2016.10.014

of using the same integration points for constructing the numerical quadrature. This approach provides in turn the luxury of evaluating quadratures for any arbitrary integration points for any desired degree of accuracy; thus increasing the accuracy of collocation schemes using relatively small number of collocation points; cf. [11,14–16].

In the current article, we extend the works of Elgindy and Smith-Miles [11,14,15], and develop some novel and efficient Gegenbauer integration matrices (GIMs) and quadratures based on the stable barycentric representation of Lagrange interpolating polynomials and the explicit barycentric weights for the GG points. The present numerical scheme represents an improvement over the aforementioned works as we reduce the computational cost and time complexity required for computing the numerical quadratures while sharing the same order of accuracy.

The rest of the article is organized as follows: In Section 2, we give some basic preliminaries relevant to Gegenbauer polynomials and their orthogonal basis and linear barycentric rational interpolations. In Section 3, we derive the barycentric GIM and quadrature, and provide a rigorous error and convergence analysis. In Section 4, we construct the optimal barycentric GIM in some optimality measure, and analyze its associated quadrature error in Section 4.1. Section 5 is devoted for a comprehensive discussion on some efficient computational algorithms required for the construction of the novel GIMs and quadratures. A discussion on how to resolve boundary-value problems using the barycentric GIM is presented in Section 5.1. Three numerical test examples are studied in Section 6 to assess the efficiency and accuracy of the numerical scheme. We provide some concluding remarks and possible future directions in Section 7. Finally, a detailed set of pseudocodes for the established computational algorithms is presented in Appendix A.

2. Preliminaries

In this section, we briefly recall some preliminary properties of the Gegenbauer polynomials and their orthogonal interpolations. The Gegenbauer polynomial $G_n^{(\alpha)}(x)$, of degree $n \in \mathbb{Z}^+$, and associated with the parameter $\alpha > -1/2$, is a real-valued function, which appears as an eigensolution to a singular Sturm–Liouville problem in the finite domain $[-1, 1]$ [27]. It is a symmetric Jacobi polynomial, $P_n^{(\nu_1,\nu_2)}(x)$, with $\nu_1 = \nu_2 = \alpha - 1/2$, and can be standardized through [16, Eq. (A.1)]. It is an odd function for odd n and an even function for even n. The Gegenbauer polynomials can be generated by the three-term recurrence equations [14, Eq. (A.4)], or in terms of the hypergeometric functions [11, Eq. (2.3)]. The weight function for the Gegenbauer polynomials is the even function $w^{(\alpha)}(x) = (1 - x^2)^{\alpha - 1/2}$. The Gegenbauer polynomials form a complete orthogonal basis polynomials in $L^2_{w^{(\alpha)}}[-1, 1]$, and their orthogonality relation is given by the following weighted inner product:

$$\left(G_m^{(\alpha)}, G_n^{(\alpha)}\right)_{w^{(\alpha)}} = \int_{-1}^{1} G_m^{(\alpha)}(x)\, G_n^{(\alpha)}(x)\, w^{(\alpha)}(x)\, dx = \left\| G_n^{(\alpha)} \right\|^2_{w^{(\alpha)}} \delta_{m,n} = \lambda_n^{(\alpha)} \delta_{m,n}, \tag{2.1}$$

where

$$\lambda_n^{(\alpha)} = \left\| G_n^{(\alpha)} \right\|^2_{w^{(\alpha)}} = \frac{2^{1-2\alpha}\,\pi\,\Gamma(n+2\alpha)}{n!\,(n+\alpha)\,\Gamma^2(\alpha)}, \tag{2.2}$$

is the normalization factor, and $\delta_{m,n}$ is the Kronecker delta function. We denote the GG nodes and their corresponding Christoffel numbers by $x_{n,k}^{(\alpha)}, \varpi_{n,k}^{(\alpha)}, k = 0, \ldots, n$, respectively. The reader may consult Refs. [1,2,10,14,27] for more information about this elegant family of polynomials.

2.1. Orthogonal Gegenbauer interpolation

The function

$$P_n f(x) = \sum_{j=0}^{n} \tilde{f}_j\, G_j^{(\alpha)}(x), \tag{2.3}$$

is the Gegenbauer interpolant of a real function f defined on $[-1, 1]$, if we compute the coefficients $\tilde{f}_j$ so that

$$P_n f(x_k) = f(x_k), \quad k = 0, \ldots, n, \tag{2.4}$$

for some nodes $x_k \in [-1, 1], k = 0, \ldots, n$. If we choose the interpolation points $x_k, k = 0, \ldots, n$, to be the GG nodes $x_{n,k}^{(\alpha)}, k = 0, \ldots, n$, then we can simply compute the discrete Gegenbauer transform using the discrete inner product created from the GG quadrature by the following formula:

$$\tilde{f}_j = \frac{\left(P_n f, G_j^{(\alpha)}\right)_n}{\left\| G_j^{(\alpha)} \right\|^2_{w^{(\alpha)}}} = \frac{\left(f, G_j^{(\alpha)}\right)_n}{\left\| G_j^{(\alpha)} \right\|^2_{w^{(\alpha)}}} = \frac{1}{\lambda_j^{(\alpha)}} \sum_{k=0}^{n} \varpi_{n,k}^{(\alpha)}\, f_{n,k}^{(\alpha)}\, G_j^{(\alpha)}\left(x_{n,k}^{(\alpha)}\right), \quad j = 0, \ldots, n, \tag{2.5}$$