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Uniform convergence of multigrid methods for adaptive meshes

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ABSTRACT

In this paper we study the multigrid methods for adaptively refined finite element meshes. In our multigrid iterations, on each level we only perform relaxation on new nodes and the old nodes whose support of nodal basis function have changed. The convergence analysis of the algorithm is based on the framework of subspace decomposition and subspace correction. In order to decompose the functions from the finest finite element space into each level, a new projection is presented in this paper. Briefly speaking, this new projection can be seemed as the weighted average of the local L^2 projection. We can perform our subspace decomposition through this new projection by its localization property. Other properties of this new projection are also presented and by these properties we prove the uniform convergence of the algorithm in both 2D and 3D. We also present some numerical examples to illustrate our conclusion.

1. Introduction

In this paper we consider multigrid methods for adaptively refined finite element meshes. We consider the following modeling problem: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product, $f \in L^2(\Omega)$ and $\Omega \in \mathbb{R}^2$ is a polygonal domain or $\Omega \in \mathbb{R}^3$ is a polyhedral domain.

Adaptive methods are widely used in the finite element methods. Through a posteriori error estimates, adaptive methods provide an efficient way to refine the meshes and increase the efficiency of computation. The convergence of the adaptive methods is studied in [5,11,17,19] and recently the optimal convergence order for adaptive methods has been proved in [8,21]. The remaining part to constitute the entire optimal computational complexity is the solving for the discrete linear system arising from the adaptive procedure.

To solve the discrete problems from the adaptive grid, multigrid methods have been widely used. [2,7] developed the multilevel adaptive technique (MLAT). The fast adaptive composite (FAC) grid method was studied in [15,16]. Multigrid methods for adaptive grid were also studied in [1,4,6]. Hierarchical basis (HB) methods [3,27] perform relaxations only at the new nodes on each level, they are most economical but the uniform convergence is not guaranteed. To achieve uniform

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convergence, more relaxations must be added. [18] studied the algorithm that relaxations are performed at the new nodes and all their neighbors (whose basis functions have a support that intersects with the support of basis functions of the new nodes). A more economical way is that relaxations are performed at the new nodes and only their immediate neighbors (whose support of nodal basis function have changed) [10,23].

[23] studied the newest vertex bisection algorithm for the 2D case, and proved the uniform convergence of the multigrid V-cycle algorithm with Gauss–Seidel relaxation performed only on new nodes and their immediate neighbors. In the analysis of [23], Xu–Zikatanov identity [26] is applied and the Scott–Zhang projection [20] is used to decompose the functions in the finite element space. In 2D cases, when we do a subtraction between two neighboring Scott–Zhang projections, the difference is a function that vanishes on all the nodes except the new nodes and their immediate neighbors. However, in 3D cases, the Scott–Zhang projection no longer has this property. Therefore the decomposition (15) can only be satisfied in 2D case when using Scott–Zhang projections. That is the reason why the conclusion in [23] is hard to be improved to the 3D cases. The most important contribution in our paper is that we propose a new local projection instead of the Scott–Zhang projection. In 3D cases, this new projection also guarantees the same decomposition (15) as in 2D. Through this new projection, we improve the conclusion in [23] to the 3D cases.

Recently, [10] proved the uniform convergence of the multigrid V-cycle algorithm of adaptive meshes in both 2D and 3D cases. They also used the Scott–Zhang projection to decompose the function in the finite element space to apply the Xu–Zikatanov identity. In their algorithm, additional smoothings on each freedom of the finest grid are needed besides the smoothings on new nodes and their immediate neighbors. For the linear elements with this method, the number of smoothings in a problem with degree of freedoms N is about $4N$. In all these $4N$ smoothings, $3N$ smoothings are on new nodes and their immediate neighbors and N smoothings are on each node of the finest grid. In this paper, we are performing relaxations only on new nodes and their immediate neighbors. Because of the absence of the smoothings on each node of the finest grid, the number of smoothings in this algorithm is at most $3N$. Furthermore, this number is usually less than $3N$ because of the repeated counting of the new nodes' neighbors. By our new projection instead of the Scott–Zhang projection, we also prove the uniform convergence of this algorithm for adaptive meshes in both 2D and 3D cases.

Similarly as in [23] and [10], our analysis of the convergence is based on the Xu–Zikatanov identity of the framework of subspace correction [24]. In the subspace decomposition, we choose the subspaces to be the node basis function spaces of the new nodes and their immediate neighbors on each level. In order to decompose the function in the entire finite element space into each subspace, we first need decompose the function into each level. A special local L^2 average projection is presented to take place of the Scott–Zhang projection in this paper, and the decomposition on each level is defined to be the difference of this projection between the neighboring two levels. The most important advantage of this projection is that the decomposition in each level only has non-zero values on new nodes and their immediate neighbors, both in 2D and 3D cases (Lemma 3.1). Then on each level we can continue decomposing the function into the node basis functions of new nodes and their immediate neighbors. Some other properties of this projection are proposed, such as the stability of the L^2 norm (Lemma 3.2, Lemma 3.3), the stability of the energy norm (Lemma 3.4, Lemma 3.5), the approximation property (Lemma 3.6) and the stability of the multilevel decomposition (Lemma 3.7). The stability of the multilevel decomposition is important for our theoretical analysis. The proof of Lemma 3.7 relies on a relation between the sequence of the adaptive meshes and a sequence of nested quasi-uniform meshes, and this method is inspired from [23]. By these properties and the similar analysis developed in [13,14], we can prove the uniform convergence of our algorithm. Furthermore, we consider the discretization of high order Lagrange element and also prove the uniform convergence through the new projection.

In this paper, we denote $x_1 \lesssim y_1$ and $x_2 \gtrsim y_2$ if there exist generic positive constants C_1 and C_2 independent of any other parameters related to the mesh, the spaces and, in particular, the coefficients such that $x_1 \leqslant C_1 y_1$ and $x_2 \geqslant C_2 y_2$. Furthermore, we denote $x_3 \simeq y_3$ if $x_3 \lesssim y_3$ and $x_3 \gtrsim y_3$. The general notations of the Sobolev space are also used. For some domain ω, the $L^2(\omega)$ norm and the $H^1(\omega)$ semi-norm are denoted by $\|\cdot\|_{0,\omega}$ and $|\cdot|_{1,\omega}$ respectively. Specially, when ω is the entire domain Ω, we may omit the tag of domain and use the notation $\|\cdot\|_0$ and $|\cdot|_1$. Additionally, we use the notation $\|\cdot\|_a = |\cdot|_1$.

The rest of this paper is organized as follows. In Section 2, we describe our adaptive meshes with some settings, and give the subspace correction algorithm. In Section 3, we introduce our new projection and prove some properties of this projection. In Section 4, we prove the uniform convergence of the multigrid methods for the adaptive meshes. In Section 5, we consider the discretization of high order Lagrange element and prove its uniform convergence. In Section 6, we present some numerical results to illustrate our theoretic results.

2. Adaptive meshes and the subspace correction algorithm

2.1. Adaptive meshes

Suppose we have a sequence of nested triangulations $\mathcal{T}_k = \{\tau_k^i\}(1 \le k \le L,\ 1 \le i \le p_k)$ of Ω, where τ_k^i is simplex. Now we denote by $V_k(\subset H_0^1(\Omega))$ the finite element space of the continuous piecewise linear functions associated with triangulation $\mathcal{T}_k$. Then

$$V_1 \subset \cdots \subset V_k \subset \cdots \subset V_L,$$