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A numerical approximation of the two-dimensional elastic wave scattering problem via integral equation method

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A R T I C L E I N F O

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A B S T R A C T

In this paper, a numerical method is proposed to approximate the solution of a two-dimensional scattering problem of time-harmonic elastic wave from a rigid obstacle. By Helmholtz decomposition, the scattering problem is reduced to a system of Helmholtz equations with coupled boundary conditions. Then, we prove that the system of Helmholtz equations has only one solution under certain conditions, and propose an integral equation method to solve it numerically based on Tikhonov regularization method. Finally, numerical examples are presented to show the feasibility and effectiveness of the proposed method.

1. Introduction

The scattering problems of elastic wave have aroused much attention from both the direct scattering problems [1,4,6,12,17] and the inverse scattering problems [2,7,9,10,13–16] for their significant applications in many scientific areas such as geophysics, seismology, nondestructive testing and geophysical exploration [12]. The direct scattering problems are to determine the elastic wave field from a knowledge of the obstacle, while the inverse scattering problems are to identify the obstacle from measurements of the far field or the near field. In this paper, we consider a direct scattering problem of time-harmonic elastic wave in an isotropic homogeneous medium at a rigid obstacle.

Let us first introduce the geometry on which the considered problems is defined. Consider a two-dimensional elastically rigid obstacle described by a bounded domain D with C^2-boundary ∂D, and suppose that its infinite exterior domain $\mathbb{R}^2 \setminus \bar{D}$ is filled with a homogeneous and isotropic elastic medium with a unit mass density. Denote by $\tau = (\tau_1, \tau_2)$ the unit tangent vector and by $\nu = (\nu_1, \nu_2)$ the outward normal vector on a closed curve such as ∂D if they do not give rise to confusion. Moreover, they meet that

$$\nu_1 = \tau_2, \quad \nu_2 = -\tau_1. \tag{1.1}$$

Let $\mathbf{u} = (u_1, u_2)$ and u be a vector function and a scalar function, respectively. The scalar curl operator and the vector curl operator are defined respectively by

$$\operatorname{curl} \mathbf{u} := \partial_x u_2 - \partial_y u_1, \quad \mathbf{curl}\, u := (\partial_y u, -\partial_x u).$$

Then, the propagation of time-harmonic wave in the exterior of the bounded domain D with Lamé constants λ and μ and density ρ is modeled by a two-dimensional Navier equation [2,13,14,16]

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$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\omega^2\mathbf{u} = 0 \ \text{ in } \mathbb{R}^2\setminus\bar{D}, \tag{1.2}$$

where $\mathbf{u} = \mathbf{u}^{in} + \mathbf{u}^{sc}$ denotes the total displacement vector field, $\mathbf{u}^{in}$ is the incident wave, $\mathbf{u}^{sc}$ is the scattered wave, ω is the angular frequency and the Lamé constants satisfy $\mu > 0$ and $\lambda + \mu > 0$. For simplicity, we assume $\rho \equiv 1$ throughout the following text. Because the obstacle is elastically rigid, it holds the homogeneous Dirichlet boundary condition

$$\mathbf{u} = 0 \ \text{ on } \partial D. \tag{1.3}$$

Given an incident field $\mathbf{u}^{in}$ such that $\mathbf{u}^{in}|_{\partial D} \in [C^{1,\sigma}(\partial D)]^2$, where $\mathbf{u}^{in}$ satisfies the Navier equation (1.2). Then, the two-dimensional scattering problem of elastic wave from a rigid obstacle is to find the scattered field $\mathbf{u}^{sc} \in [C^2(\mathbb{R}^2\setminus\bar{D}) \cap C^1(\mathbb{R}^2\setminus D)]^2$ such that

$$\mu\Delta\mathbf{u}^{sc} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u}^{sc} + \omega^2\mathbf{u}^{sc} = 0 \ \text{ in } \mathbb{R}^2\setminus\bar{D}, \tag{1.4}$$

$$\mathbf{u}^{sc} = -\mathbf{u}^{in} \ \text{ on } \partial D, \tag{1.5}$$

$$\hat{x}\cdot\nabla\nabla\cdot\mathbf{u}^{sc} - ik_p\nabla\cdot\mathbf{u}^{sc} = o(\frac{1}{|x|}), \ \ |x|\to\infty, \tag{1.6}$$

$$\hat{x}\cdot\nabla\mathrm{curl}\,\mathbf{u}^{sc} - ik_s\mathrm{curl}\,\mathbf{u}^{sc} = o(\frac{1}{|x|}), \ \ |x|\to\infty, \tag{1.7}$$

where $\hat{x} := \frac{x}{|x|}$, $k_p = \frac{\omega}{\sqrt{\lambda+2\mu}}$ is the compressional wavenumber, $k_s = \frac{\omega}{\sqrt{\mu}}$ is the shear wavenumber. (1.6) and (1.7) are the Kupradze radiation conditions which guarantee that the scattered waves are outgoing. As we all know, this elastic scattering problem has at most one solution; and the existence of its solution is proved by using the integral equation method [7,17]. We refer to [5,9] for solving numerically the direct scattered problem (1.4)–(1.7) by using the integral equation method with Green's tensor $\Gamma(x,y)$ of the Navier equation, where the scattered field is assumed that in the form

$$\mathbf{u}^{sc}(x) = \int_{\partial D}\Gamma(x,y)\varphi(y)ds(y), \ x\in\mathbb{R}^2\setminus\bar{D}.$$

Next we state the uniqueness and existence of the elastic scattering solution as the following lemma.

Lemma 1. *[7,17] For a given incident field $\mathbf{u}^{in}$ such that $\mathbf{u}^{in}|_{\partial D} \in [C^{1,\sigma}(\partial D)]^2$, the elastic scattering problem (1.4)–(1.7) has a unique solution in $[C^2(\mathbb{R}^2\setminus\bar{D}) \cap C^1(\mathbb{R}^2\setminus D)]^2$.*

The rest of this paper is organized as follows. In section 2, we first formulate the elastic scattering problem into a system of Helmholtz equations with coupled boundary conditions by Helmholtz decomposition. Further, we prove that the system of Helmholtz equations has only one solution. In section 3, an integral equation method is proposed to solve the system of Helmholtz equations based on Tikhonov regularization method. In section 4 numerical examples are presented to show the validity and efficiency of the proposed method. Finally, some conclusions are drawn and further studies are discussed.

2. Helmholtz equations with coupled boundary conditions

It is valuable to be noted that the two components of $\mathbf{u}^{sc}$ are coupled in the Navier equation (1.4). To obtain decoupled equations, it is crucial to introduce the Helmholtz decomposition [13,14,16] for splitting the total field into a compressional part and a shear part. For any solution $\mathbf{u}^{sc}$ of equation (1.4), the Helmholtz decomposition takes the form

$$\mathbf{u}^{sc} = \nabla\phi + \mathbf{curl}\psi, \tag{2.1}$$

where ϕ and ψ are scalar potential functions.

Substituting (2.1) into (1.4) yields

$$\nabla[(\lambda+2\mu)\Delta\phi + \omega^2\phi] + \mathbf{curl}[\mu\Delta\psi + \omega^2\psi] = 0,$$

which is fulfilled if ϕ and ψ satisfy the following Helmholtz equations:

$$\Delta\phi + k_p^2\phi = 0, \ \ \Delta\psi + k_s^2\psi = 0, \tag{2.2}$$

where $k_p = \frac{\omega}{\sqrt{\lambda+2\mu}}$ and $k_s = \frac{\omega}{\sqrt{\mu}}$. Combining (2.1) with (2.2), we have explicit representations of ϕ and ψ in terms of $\mathbf{u}^{sc}$:

$$\phi = -\frac{1}{k_p^2}\nabla\cdot\mathbf{u}^{sc}, \ \ \psi = \frac{1}{k_s^2}\mathrm{curl}\,\mathbf{u}^{sc}. \tag{2.3}$$

So, ϕ is the compressional part and ψ is the shear part. Correspondingly, we call k_p and k_s the compressional wavenumber and the shear wavenumber, respectively.