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On symmetric graphs of order four times an odd square-free integer and valency seven ☆

Jiangmin Pan [a,*], Bo Ling [b], Suyun Ding [c]

[a] School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan, PR China
[b] School of Mathematics and Computer Science, Yunnan Minzu University, Kunming, Yunnan, PR China
[c] School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan, PR China

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ABSTRACT

A graph is called *symmetric* if its automorphism group is transitive on its arcs. In this paper, we classify symmetric graphs of order four times an odd square-free integer and valency seven. It is shown that, either the graph is isomorphic to one of 9 specific graphs or its full automorphism group is isomorphic to $\mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$ or $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$ with $p\equiv\pm1(\mathrm{mod}\ 7)$ a prime.

1. Introduction

For a simple, connected and undirected graph Γ, denote by $V\Gamma$ its vertex set, and denote by $\Gamma(\alpha)=\{\beta\in V\Gamma\mid \beta \text{ is adjacent to } \alpha\}$, the neighbor set of a vertex $\alpha\in V\Gamma$. The size $|V\Gamma|$ is called the *order* of Γ. If Γ is a regular graph (each vertex of Γ has the same number of adjacent vertices), then the size $|\Gamma(\alpha)|$ is called the *valency* (or *degree*) of Γ. For adjacent vertices α and β, the ordered pair (α,β) is called an *arc* of Γ, and the unordered pair $\{\alpha,\beta\}$ is called an *edge* of Γ, namely, the edge $\{\alpha,\beta\}$ corresponds to two arcs (α,β) and (β,α). The set of edges and the set of arcs of Γ are denoted by $E\Gamma$ and $A\Gamma$, respectively. Then Γ is called *vertex-transitive*, *edge-transitive* or *arc-transitive*, if the full automorphism group $\mathrm{Aut}\Gamma$ of Γ is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$, respectively. An arc-transitive graph is also called a *symmetric graph*.

Let s be a positive integer. An s-*arc* of Γ is a sequence $\alpha_0,\alpha_1,\ldots,\alpha_s$ of $s+1$ vertices such that α_{i-1},α_i are adjacent for $1\le i\le s$ and $\alpha_{i-1}\ne\alpha_{i+1}$ for $1\le i\le s-1$. Then Γ is called (G,s)-*arc-transitive* if G is transitive on the set of s-arcs of Γ. In particular, if $\mathrm{Aut}\Gamma$ is regular on $\mathrm{A}\Gamma$, then Γ is called *one-regular*; if $\mathrm{Aut}\Gamma$ is transitive on the set of s-arcs but not transitive on the set of $(s+1)$-arcs of Γ, then Γ is called s-*transitive*.

Characterizing transitive graphs was initiated by a remarkable result of Tutte (1949) which says there exists no finite s-arc-transitive cubic graph for $s\ge 6$. This result was extended by Weiss [35] who proved there is no 8-transitive graph of valency at least 3. Since then, studying certain families of transitive graphs has been one of the most important topics in algebraic graph theory; see for example [17,29] and references therein.

A positive integer is called *square-free* if it is not divisible by any prime square. As a natural step, studying transitive graphs of order small number times a square-free integer has received quite a lot attention in the literature. Chao [3] classified symmetric graphs of prime order, and Cheng and Oxley [4] classified edge-transitive graph of order twice a prime. These

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* Corresponding author.
E-mail addresses: jmpan@ynu.edu.cn (J. Pan), bolinggxu@163.com (B. Ling).

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Table 1
'Sporadic' 7-valent symmetric graphs of four times an odd square-free order.

Row	Γ	n	$\mathrm{Aut}\Gamma$	$(\mathrm{Aut}\Gamma)_\alpha$	Transitivity	Remark
1	$\mathcal{C}_{60}$	$3\cdot 5$	S_7	F_{42}	$2-$ transitive	not bipartite
2	$\mathcal{C}_{660}$	$3\cdot 5\cdot 11$	$(\mathbb{Z}_2\times \mathrm{M}_{22}).\mathbb{Z}_2$	$\mathrm{ASL}(3,2)\times\mathbb{Z}_2$	$2-$ transitive	bipartite
3	$\mathcal{C}_{1716}$	$3\cdot 11\cdot 13$	S_{13}	$\mathrm{S}_7\times\mathrm{S}_6$	$3-$ transitive	not bipartite
4	$\mathcal{C}_{4180}$	$5\cdot 11\cdot 19$	J_1	F_{42}	$2-$ transitive	not bipartite
5	$\mathcal{C}^1_{12540}$	$3\cdot 5\cdot 11\cdot 19$	J_1	F_{14}	$1-$ transitive	not bipartite
6	$\mathcal{C}^2_{12540}$	$3\cdot 5\cdot 11\cdot 19$	J_1	F_{14}	$1-$ transitive	not bipartite
7	$\mathcal{C}^3_{12540}$	$3\cdot 5\cdot 11\cdot 19$	J_1	F_{14}	$1-$ transitive	not bipartite
8	$\mathcal{C}^4_{12540}$	$3\cdot 5\cdot 11\cdot 19$	J_1	F_{14}	$1-$ transitive	not bipartite
9	$\mathcal{C}^5_{12540}$	$3\cdot 5\cdot 11\cdot 19$	J_1	F_{14}	$1-$ transitive	not bipartite

results were generalized to the order of any two distinct primes case by Praeger et al. [30,31], and generalized to the vertex-transitive case by Marušič and Scapellato [26]. Also, Feng [9] determined all arc-regular prime-valent graphs of square-free order by proving that such graphs are normal Cayley graphs of dihedral groups, and the result was extended to Cayley graphs of any dihedral groups by [27]. More recently, Li et al. [20] characterized 'basic' edge-transitive graphs (that is, each normal subgroup of edge-transitive automorphism groups has at most two orbits on the vertex set) of square-free order, with many cases needing further research. It seems quite difficult to approach a general classification of transitive graphs of square-free order. For small valency case, a lot of results have been obtained in recent years. For example, see [19,25] for vertex-transitive and edge-transitive cubic graphs of square-free order, see [18,21] for edge-transitive tetravalent graphs of square-free order, and see [7,22] for symmetric graphs of square-free order and valencies 5,6 and 7. Moreover, [24] classified cubic symmetric graphs of order four times an odd square-free integer, and [14,36] determined 4-valent and 7-valent symmetric graph of order $4p$ with p an odd prime. The main purpose of this paper is to extend the result in [14] by classifying 7-valent symmetric graphs of order $4n$ with n an arbitrary odd square-free integer.

The main result of this paper is the following theorem, where the graphs appearing in Table 1 are introduced in Section 2, and the notations used are standard, refer to [5].

Theorem 1.1. *Let Γ be a 7-valent symmetric graph of order $4n$, where n is an odd square-free integer. Then one of the following statements holds:*

(1) *The triple $(\Gamma, n, \mathrm{Aut}\Gamma)$ is listed in Table 1.*
(2) $\mathrm{Aut}\Gamma\cong\mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$ *or* $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$, *where $p\equiv\pm1(\mathrm{mod}\ 7)$ is a prime.*

In particular, if $|V\Gamma|>12540$, then Γ satisfies part (2).

We remark that it seems not feasible to determine all the possible values of p in part (2) for general square-free integer n. However, if the number of the prime divisors of n is fixed, then it is not difficult to determine the possible values of p and hence all corresponding graphs Γ.

After this introductory section, some preliminary results and examples that appear in Theorem 1.1 are introduced in Section 2. Then, we consider the 'trivial soluble radical case' in Section 3 and 'nontrivial soluble radical case' in Section 4 (recall the soluble radical of a group is its largest soluble normal subgroup).

2. Preliminary results and examples

2.1. Preliminary results

In this subsection, we quote some preliminary results that will be used in the subsequent sections.

For a group G, the *Fitting subgroup* of G is the largest nilpotent normal subgroup of G. Clearly, the Fitting subgroup is a characteristic subgroup. The following is a basic property of the Fitting subgroup of soluble groups.

Lemma 2.1 (*[34, P. 30, Corollary]*)**.** *Let G be a soluble group and F the Fitting subgroup of G. Then $\mathrm{F}\neq 1$ and the centralizer $C_G(\mathrm{F})\le\mathrm{F}$.*

The maximal subgroups of $\mathrm{PSL}(2,q)$ are known, see [6, Section 239].

Lemma 2.2. *Let $T=\mathrm{PSL}(2,q)$ and H a maximal subgroup of T, where $q=p^n\ge5$ with p a prime. Then one of the following statements holds, where $d=(2,q-1)$.*

(1) $H\in\{\mathrm{A}_4,\mathrm{S}_4,\mathrm{A}_5,\mathrm{D}_{2(q-1)/d},\mathrm{D}_{2(q+1)/d},\mathbb{Z}_p^n:\mathbb{Z}_{(q-1)/d}\}$;
(2) $H\cong\mathrm{PSL}(2,p^m)$ *with n/m an odd integer;*
(3) $H\cong\mathrm{PGL}(2,p^{n/2})$ *with n an even integer.*