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Circumferences of 3-connected claw-free graphs, II

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ABSTRACT

For a graph H, the circumference of H, denoted by $c(H)$, is the length of a longest cycle in H. It is proved in Chen (2016) that if H is a 3-connected claw-free graph of order n with $\delta \geq 8$, then $c(H) \geq \min\{9\delta - 3, n\}$. In Li (2006), Li conjectured that every 3-connected k-regular claw-free graph H of order n has $c(H) \geq \min\{10k - 4, n\}$. Later, Li posed an open problem in Li (2008): how long is the best possible circumference for a 3-connected regular claw-free graph? In this paper, we study the circumference of 3-connected claw-free graphs without the restriction on regularity and provide a solution to the conjecture and the open problem above. We determine five families $\mathcal{F}_i$ $(1 \leq i \leq 5)$ of 3-connected claw-free graphs which are characterized by graphs contractible to the Petersen graph and show that if H is a 3-connected claw-free graph of order n with $\delta \geq 16$, then one of the following holds:

(a) either $c(H) \geq \min\{10\delta - 3, n\}$ or $H \in \mathcal{F}_1$.
(b) either $c(H) \geq \min\{11\delta - 7, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2$.
(c) either $c(H) \geq \min\{11\delta - 3, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.
(d) either $c(H) \geq \min\{12\delta - 10, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$.
(e) if $\delta \geq 23$ then either $c(H) \geq \min\{12\delta - 7, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$.

This is also an improvement of the prior results in Chen (2016), Lai et al. (2016), Li et al. (2009) and Mathews and Sumner (1985).

1. Introduction

Graphs considered in this paper are finite and loopless. A graph is called a *multigraph* if it contains multiple edges. A graph without multiple edges is called a *simple graph* or simply a graph. As in [1], $\alpha'(G)$, $\kappa'(G)$ and $d_G(v)$ denote the size of a maximum matching in G, the edge-connectivity of G and the degree of a vertex v in G, respectively. The minimum degree of a graph G is denoted by $\delta(G)$ or δ. For a vertex $v \in V(G)$, let $E_G(v)$ be the set of edges in G incident with v. Thus, when G is a simple graph, $|E_G(v)| = d_G(v)$. An edge cut X of a graph G is *essential* if each of the components of $G - X$ contains an edge. A graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut of size less than k. A vertex set $U \subseteq V(G)$ is called a *covering* of G if every edge of G is incident with a vertex in U. The minimum number of vertices in a covering of G is called the *covering number* of G and denoted by $\beta(G)$. An edge $e = uv$ is called a *pendant edge* if $\min\{d_G(u), d_G(v)\} = 1$.

A *trail* T is a finite sequence $T = u_0e_1u_1e_2u_2 \cdots e_ru_r$, whose terms are alternately vertices and edges, with $e_i = u_{i-1}u_i$ $(1 \leq i \leq r)$, where the edges are distinct. A trail T is a *closed trail* if $u_0 = u_r$ and is called a (u, v)-trail if $u = u_0$ and $v = u_r$. A trail or closed trail T in a graph G is called a *spanning trail* (ST) or a *spanning closed trail* (SCT) of G if $V(G) = V(T)$ and is called a *dominating trail* (DT) or a *dominating closed trail* (DCT) if $E(G - V(T)) = \emptyset$. The family of graphs with SCTs is denoted by $\mathcal{SL}$. A graph G is called a *DCT graph* if G has a DCT.

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The circumference of a graph H, denoted by $c(H)$, is the length of a longest cycle in H. A graph H is *claw-free* if H does not contain an induced subgraph isomorphic to $K_{1,3}$. In this paper, we will be concerned with the circumference of 3-connected claw-free graphs.

In [14], Matthews and Sumner proved that every 2-connected claw-free graph H of order n has $c(H) \geq \min\{n, 2\delta + 4\}$. Li, et al. [13] proved that every 3-connected claw-free graph H of order n has $c(H) \geq \min\{n, 6\delta - 15\}$. Solving a conjecture posed in [13], we proved the following.

Theorem 1.1 (*[6]*). *If H is a 3-connected claw-free graph of order n and $\delta \geq 8$, $c(H) \geq \min\{n, 9\delta - 3\}$.*

Theorem 1.1 is best possible in the sense that if $H_r = L(G_r)$ where G_r is obtained from the Petersen graph P by adding $r > 0$ pendant edges at each vertex of P, then $c(H_r) = 9\delta(H_r) - 3$.

For regular claw-free graphs, Li posed the following conjecture in [11].

Conjecture 1.2 (*Li, Conjecture 6 [11]*). *Every 3-connected k-regular claw-free graph H on n vertices has $c(H) \geq \min\{10k - 4, n\}$.*

In [12], Li restated the conjecture with a different lower bound on $c(H)$.

Conjecture 1.3 (*Li, Conjecture 5.17 [12]*). *Every 3-connected k-regular claw-free graph H on n vertices has $c(H) \geq \min\{12k - 7, n\}$.*

It was stated in [12] that Conjecture 1.3 was from [11]. However, Conjecture 1.2 is the only conjecture in [11]. We do not know why "$10k - 4$" is changed to "$12k - 7$" in Conjecture 1.3. Maybe it is more proper to treat them as open problems. In fact, Li posed an open problem in [12].

Problem 1.4 (*Li, Problem 5.18 [12]*). How long is the best possible circumference for a 3-connected regular claw-free graph?

Note that H_r mentioned above is a non-regular claw-free graph. These conjectures and the open problem suggest a more general problem: how long is the best possible circumference for a 3-connected claw-free graph H if $H \neq H_r$?

In this paper, using much improved techniques employed in [6], we provide solutions to these open problems and conjectures. Our results are given in next section.

2. Main results and Ryjáček's closure concept

For a graph G, the line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. As we know that all line graphs are claw-free and a connected line graph $H \neq K_3$ has a unique graph G with $H = L(G)$. We call G the preimage graph of H. Ryjáček [16] defined the closure $cl(H)$ of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H as long as this is possible, and H is said to be *closed* if $H = cl(H)$.

Theorem 2.1. *(Ryjáček [16]). Let H be a claw-free graph and $cl(H)$ its closure. Then*
(a) *$cl(H)$ is well defined, and $\kappa(cl(H)) \geq \kappa(H)$;*
(b) *there is a K_3-free simple graph G such that $cl(H) = L(G)$;*
(c) *for every cycle C_0 in $L(G)$, there exists a cycle C in H with $V(C_0) \subseteq V(C)$.*

Let P be the Petersen graph. Let Φ_a and Φ_b be two connected K_3-free simple graphs. Let $P(\Phi_a, \Phi_b)$ be an essentially 3-edge-connected K_3-free simple graph obtained from P by replacing a vertex v_a in P by Φ_a and replacing a vertex v_b in P by Φ_b, and by adding at least $r > 0$ pendant edges at each vertex of $V(P) - \{v_a, v_b\}$ and subdividing m edges of P for $m = 0, 1, \ldots, 15$.

Let Π_a and Π_b be two families of K_3-free graphs. Define $\mathcal{P}(\Pi_a, \Pi_b)$ be the family of graphs below:
$\mathcal{P}(\Pi_a, \Pi_b) = \{G \mid G = P(\Phi_a, \Phi_b)$ where $\Phi_a \in \Pi_a$ and $\Phi_b \in \Pi_b\}$ (see Fig. 2.1. for examples).
Here is a list of families of K_3-free graphs that will be used for Π_a or Π_b.

- Let $\mathcal{K}_{1,r}$ be the family of stars $K_{1,r}$ with $r \geq 1$ edges.
- Let $\mathcal{K}_{2,r}$ be the family of spanning connected subgraphs of $K_{2,r}$ for some $r \geq 2$.
- Let $\mathcal{Q}_t$ be the family of K_3-free connected simple graphs G with $\alpha'(G) = t$.

Note that $K_{t,s} \in \mathcal{Q}_t$ for $t \leq s$ and $\mathcal{K}_{t,s} = \mathcal{Q}_t$ for $t \in \{1, 2\}$ and $s \geq t$ (see Proposition 3.3).
For essentially 3-edge-connected K_3-free simple graphs, we define the following families:

- $\mathcal{P}_1 = \mathcal{P}(\mathcal{K}_{1,r}, \mathcal{K}_{1,r})$.
- $\mathcal{P}_2 = \mathcal{P}(\mathcal{K}_{2,r}, \mathcal{K}_{1,r})$.
- $\mathcal{P}_3 = \mathcal{P}(\mathcal{Q}_3, \mathcal{K}_{1,r})$.
- $\mathcal{P}_4 = \mathcal{P}(\mathcal{K}_{2,r}, \mathcal{K}_{2,r})$.
- $\mathcal{P}_5 = \mathcal{P}(\mathcal{Q}_4, \mathcal{K}_{1,r})$.
- $\mathcal{P}_6 = \mathcal{P}(\mathcal{Q}_3, \mathcal{K}_{2,r})$.

For each i ($1 \leq i \leq 6$), we define a family $\mathcal{F}_i$ of 3-connected claw-free graphs according to $\mathcal{P}_i$:
$\mathcal{F}_i = \{H : H$ is a 3-connected claw-free graph with $cl(H) = L(G)$ and $G \in \mathcal{P}_i\}$.
Here is our main result.