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Every planar graph without cycles of length 4 or 9 is (1, 1, 0)-colorable[☆]

Lifeng Dai, Yingqian Wang *, Jinghan Xu

Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, China

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ABSTRACT

Let $d_1, d_2, \dots, d_k$ be k non-negative integers. A graph G is $(d_1, d_2, \dots, d_k)$-colorable, if the vertex set of G can be partitioned into subsets $V_1, V_2, \dots, V_k$ such that the subgraph $G[V_i]$ induced by V_i has maximum degree at most d_i for $i = 1, 2, \dots, k$. In this paper, we prove that every planar graph without cycles of length 4 or 9 is (1, 1, 0)-colorable.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let $d_1, d_2, \dots, d_k$ be k nonnegative integers. A $(d_1, d_2, \dots, d_k)$*-coloring* of a graph $G = (V, E)$ is a mapping $\phi : V \longrightarrow \{1, \dots, k\}$ such that the subgraph $G[V_i]$ induced by V_i has maximum degree at most d_i, where $V_i = \{v \in V | \phi(v) = i\}$. G is $(d_1, d_2, \dots, d_k)$*-colorable* if it admits a $(d_1, d_2, \dots, d_k)$-coloring. The Four Color Theorem [1,2] says that every planar graph is (0, 0, 0, 0)-colorable; and the Three Color Theorem [8] says that every triangle-free planar graph is (0, 0, 0)-colorable. Assuming that only three colors are allowed to color the planar graphs, Cowen, Cowen and Woodall [6] showed that every planar graph is (2, 2, 2)-colorable; Xu [18] showed that every planar graph with neither adjacent triangles nor 5-cycles is (1, 1, 1)-colorable.

Let $\mathscr{F}_k$ be the family of the planar graphs without cycles of length 4 or k ($k \geq 5$). A central conjecture on 3-colorability of planar graphs, proposed by Steinberg in 1976 [14], states that everyone in $\mathscr{F}_5$ is (0, 0, 0)-colorable. Motivated by the Steinberg's conjecture, Lih et al. [12] showed that everyone in $\mathscr{F}_k$, $k \in \{5, 6, 7\}$, is list (1, 1, 1)-colorable; and Dong and Xu [7] showed that the same is true for $k \in \{8, 9\}$. Later, Wang and Xu [16] improved these results to that every planar graph without cycles of length 4 is list (1, 1, 1)-colorable. Also motivated by the Steinberg's conjecture, Chang et al. [3] showed that everyone in $\mathscr{F}_5$ is (4, 0, 0)- and (2, 1, 0)-colorable. Later, Hill et al., [9] showed that everyone in $\mathscr{F}_5$ is (3, 0, 0)-colorable; and Hill and Yu [10] and independently Xu, Miao and Wang [20] proved that everyone in $\mathscr{F}_5$ is (1, 1, 0)-colorable. Recently, Chen et al. [4] proved that everyone in $\mathscr{F}_5$ is (2, 0, 0)-colorable.

In 2014, Wang and Xu [15] proposed a generalized Steinberg's conjecture as follows.

A Generalized Steinberg's Conjecture: *For every* $k \in \{5, 6, 7, 8, 9\}$, *everyone in* $\mathscr{F}_k$ *is* (0, 0, 0)*-colorable.*

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* Corresponding author.
E-mail address: yqwang@zjnu.cn (Y. Wang).

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Surprisingly, Cohen-Addad et al. [5] disproved the case $k = 5$, i.e., Steinberg's conjecture is false. However, it is unknown if any other case of the generalized Steinberg's Conjecture is false. Below are some results supporting that the other cases in the generalized Steinberg's conjecture might be true.

Theorem A. (1) *Every graph in* $\mathscr{F}_6$ *is* $(3, 0, 0)$-, $(1, 1, 0)$-, $(2, 0, 0)$-*colorable, see* [17,19]*;*
(2) *Every graph in* $\mathscr{F}_7$ *is* $(3, 0, 0)$-, $(1, 1, 0)$-, $(2, 0, 0)$-*colorable, see* [11,13,15]*;*
(3) *Every graph in* $\mathscr{F}_8$ *is* $(1, 1, 0)$-*colorable, see* [15]. □

In this paper, we show the following result.

Theorem 1. *Every graph in* $\mathscr{F}_9$ *is* $(1, 1, 0)$-*colorable.* □

Together with previous results, Theorem 1 completes a stage conclusion as follows.

Theorem 2. *For every* $k \in \{5, 6, 7, 8, 9\}$*, every graph in* $\mathscr{F}_k$ *is* $(1, 1, 0)$-*colorable.* □

Thus, Theorem 2 leaves us an appealing problem as follows:

Problem 1. Is every graph in $\mathscr{F}_k$, $k \in \{5, 6, 7, 8, 9\}$, $(1, 0, 0)$-colorable? □

We strongly believe that every instance of Problem 1 has a positive answer.

The rest of this section is devoted to some definitions. Call a graph G *planar* if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a *plane* graph. For a plane graph G, we use V, E, F, and δ to denote its vertex set, edge set, face set, and minimum degree, respectively. For a vertex $v \in V$, the degree of v in G, denoted $d_G(v)$, or simply $d(v)$, is the number of edges incident with v in G. The neighborhood of v in G, denoted $N_G(v)$, or simply $N(v)$, consists of all vertices adjacent to v in G. Call v a *k-vertex*, or a *k^+-vertex*, or a *k^--vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. An edge $xy \in E$ is called a *$(d(x), d(y))$-edge*, and x is called a *$d(x)$-neighbor* of y. For a face $f \in F$, the number of steps of the boundary of f, denoted $d(f)$, is called the *degree* of f. Call f a *k-face*, or a *k^+-face*, or a *k^--face* if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. We write $f = [v_1v_2 \dots v_k]$ if $v_1, v_2, \dots, v_k$ are consecutive vertices on f in a cyclic order, and say that f is a *$(d(v_1), d(v_2), \dots, d(v_k))$-face*. A *k-cycle* is a cycle of length k. In this paper, a *triangle* is the boundary of a 3-face. Call a vertex or an edge *triangular* if it is incident with a triangle. Call a vertex u an *isolated neighbor* of v if $uv \in E$, and uv is not triangular. *A pendent* 3-*face* of a vertex v is a 3-face which does not contain v but contains a 3-vertex adjacent to v. If a 3-vertex v is incident with a 3-face, then its neighbor not incident with this 3-face is called its *outer neighbor*. If the outer neighbor of v is a k-vertex, then we call it an *outer k-neighbor* of v. Finally, let C be a cycle in G. The set of vertices lying strictly inside or outside C is denoted by $int(C)$ or $ext(C)$, respectively. If both $int(C)$ and $ext(C)$ are not empty, then C is called a *separating* cycle of G. Note that $Ext(C) \cap int(C) = \emptyset$ where $Ext(C) = V(C) \cup ext(C)$.

2. Reducibility

As usual, to *properly* color a vertex v means to assign v a color which has not been assigned to any neighbor of v. To $(1, 1, 0)$-*color*, in short, to *color*, a vertex v means to properly color v with 3, or, for $i \in \{1, 2\}$, assigns v with i if v has at most one neighbor colored i and the neighbor of v colored i, if any, is properly colored before coloring v. A *partial* $(1, 1, 0)$-coloring of G is a $(1, 1, 0)$-coloring of a vertex induced subgraph of G. If ϕ is a partial $(1, 1, 0)$-coloring of G and A a set of colored vertices in ϕ, then we define $\phi(A) = \{\phi(a) | a \in A\}$. Note that $\phi(A)$ may be a multi-set of colors.

Suppose Theorem 1 is false. Let $G = (V, E)$ be a counterexample to Theorem 1 with the fewest vertices. Clearly G is connected. Embedding G into the plane, we get a plane graph $G = (V, E, F)$. Since G has no 4-cycle, G has no adjacent triangles. Below are structural properties of G. Some of them have been obtained in some earlier papers. Here we reprove them for self-containment.

Lemma 1. $\delta(G) \geq 3$.

Proof. Suppose to the contrary that $\delta(G) \leq 2$. Let v be a vertex of degree at most 2 in G. By the minimality of G, $G' = G - v$ admits a $(1, 1, 0)$-coloring ϕ. Since $d(v) \leq 2$, we can properly color v with a color in $\{1, 2, 3\}$ that has not been assigned to any neighbor of v, giving a $(1, 1, 0)$-coloring of G, a contradiction. □

2.1. Structure involving 3-vertices

Lemma 2. *A* 3-*vertex* v *in* G *has at most one* 3-*neighbor.*

Proof. Suppose v has two 3-neighbors, say v_1 and v_2. Let v_3 be the remaining neighbor of v. By the minimality of G, $G' = G - \{v, v_1, v_2\}$ admits a $(1, 1, 0)$-coloring ϕ. Clearly we can properly color v_1 and v_2 in turn. We may assume that $\phi(N(v)) = \{1, 2, 3\}$ since otherwise we could properly color v. Now we can color v with a color in $\{1, 2\} \setminus \{\phi(v_3)\}$, giving a $(1, 1, 0)$-coloring of G, a contradiction. □