Discrete Mathematics 340 (2017) 2123–2129

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Enumerations of vertices among all rooted ordered trees with levels and degrees

Sen-Peng Eu [a], Seunghyun Seo [b], Heesung Shin [c,*]

[a] Department of Mathematics, National Taiwan Normal University, Taipei 116, Taiwan, ROC
[b] Department of Mathematics Education, Kangwon National University, Chuncheon 24341, Republic of Korea
[c] Department of Mathematics, Inha University, Incheon 22212, Republic of Korea

ARTICLE INFO

Article history:
Received 15 November 2016
Received in revised form 4 April 2017
Accepted 7 April 2017
Available online 15 May 2017

Keywords:
Enumeration
Rooted ordered tree
Degree
Level

ABSTRACT

In this paper we enumerate and give bijections for the following four sets of vertices among rooted ordered trees of a fixed size: (i) first-children of degree k at level ℓ, (ii) non-first-children of degree k at level $\ell - 1$, (iii) leaves having $k - 1$ elder siblings at level ℓ, and (iv) non-leaves of outdegree k at level $\ell - 1$. Our results unite and generalize several previous works in the literature.

1. Introduction

Let $\mathcal{T}_n$ be the set of *rooted ordered trees* with n edges. It is well known that the cardinality of $\mathcal{T}_n$ is the *nth Catalan number*

$$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}.$$

For example, there are 5 rooted ordered trees with 3 edges, see Fig. 1. Clearly the number of vertices among trees in $\mathcal{T}_n$ is

$$(n+1)\mathrm{Cat}_n = \binom{2n}{n}.$$

Given a rooted ordered tree, a vertex v is a *child* of a vertex u and u is the *parent* of v if v is directly connected to u when moving away from the root. A vertex without children is called a *leaf*. Note that by definition the root is not a child. Vertices with the same parent are called *siblings*. Since siblings are linearly ordered, when drawing trees the siblings are put in the left-to-right order. Siblings to the left of v are called *elder* siblings of v. The leftmost vertex among siblings is called the *first-child*. In Fig. 1, there are 10 first-children as well as 10 leaves, which is precisely a half of all 20 vertices in trees in $\mathcal{T}_3$.

In 1999, Shapiro [6] proved the following using generating functions.

Theorem 1. *For any positive integer n, the following four sets of vertices among trees in $\mathcal{T}_n$ are equinumerous:*

(i) *first-children,*
(ii) *non-first-children,*
(iii) *leaves, and*
(iv) *non-leaves.*

* Corresponding author.
E-mail addresses: speu@math.ntnu.edu.tw (S.-P. Eu), shyunseo@kangwon.ac.kr (S. Seo), shin@inha.ac.kr (H. Shin).

http://dx.doi.org/10.1016/j.disc.2017.04.007
0012-365X/

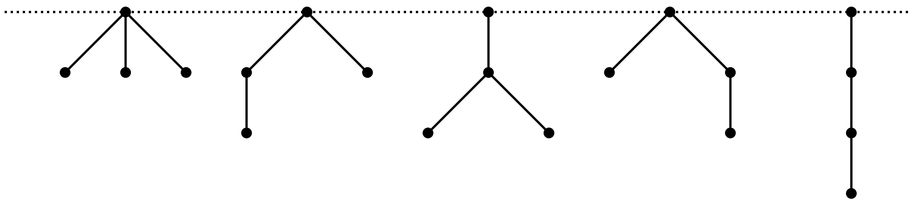

Fig. 1. The five trees in $\mathcal{T}_3$.

Here, the cardinality of each set is

$$\frac{1}{2}\binom{2n}{n},$$

which is a half of the number of vertices among trees in $\mathcal{T}_n$.

Seo and Shin [5] gave an involution proving (iii) and (iv) are equinumerous. The equality with the other two sets is easily seen: (i) and (iv) are equinumerous since each non-leaf has its unique first-child and the union of (i) and (ii) is the same with (iii) and (iv).

1.1. Degree and outdegree

The *degree* of a vertex is the number of edges incident to it. As every edge has a natural outward direction away from the root, we can have the notion of the *outdegree* of a vertex v, which is the number of edges starting at v and pointing away from the root. Since each vertex has a degree and each non-leaf has a positive outdegree, Theorem 1′ can be restated as follows.

Theorem 1′ ([6]). *Let $n \geq 1$. Among trees in $\mathcal{T}_n$, the number of vertices of positive degree equals twice the number of vertices of positive outdegree.*

In 2004, Eu, Liu, and Yeh [4] proved combinatorially the following by constructing a two-to-one correspondence which answered a problem posed by Deutsch and Shapiro [3, p. 259].

Theorem 2 ([4]). *Let $n \geq 1$. Among trees in $\mathcal{T}_n$, the number of vertices of odd degree equals twice the number of vertices of odd positive outdegree.*

In light of these two results, it is natural to ask if more can be said. In Corollary 6 we will prove that for any positive integer k the number of vertices of degree k always equals *twice* the number of that of outdegree k among trees in $\mathcal{T}_n$.

1.2. Even and odd level

A vertex v in a rooted tree T is at *level* ℓ if the distance (number of edges) from the root to v is ℓ. By an involution, the following result was obtained by Chen, Li, and Shapiro [1].

Theorem 3 ([1]). *The number of vertices at odd levels equals the number of vertices at even levels among trees in $\mathcal{T}_n$.*

It is again natural to ask if more can be said. In Corollary 8 of this paper we will prove that for any positive integer k the number of vertices of degree k at odd levels always equals the number of vertices of degree k at even levels among trees in $\mathcal{T}_n$.

1.3. Main result

In this paper we generalize the above by regarding both degrees and levels simultaneously. We will give a combinatorial proof for the following main result.

Theorem 4. *Given $n \geq 1$, for any two positive integers k and ℓ, there are one-to-one correspondences between the following four sets of vertices among trees in $\mathcal{T}_n$:*

(i) *first-children of degree k at level ℓ,*
(ii) *non-first-children of degree k at level $(\ell - 1)$,*
(iii) *leaves having exactly $(k - 1)$ elder siblings at level ℓ, and*
(iv) *non-leaves of outdegree k at level $(\ell - 1)$.*