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Improving bounds on the diameter of a polyhedron in high dimensions

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ABSTRACT

In 1992, Kalai and Kleitman proved that the diameter of a d-dimensional polyhedron with n facets is at most $n^{2+\log_2 d}$. In 2014, Todd improved the Kalai–Kleitman bound to $(n-d)^{\log_2 d}$. We improve the Todd bound to $(n-d)^{-1+\log_2 d}$ for $n \geq d \geq 7$, $(n-d)^{-2+\log_2 d}$ for $n \geq d \geq 37$, and $(n-d)^{-3+\log_2 d+\mathcal{O}(1/d)}$ for $n \geq d \geq 1$.

1. Introduction

The *diameter* $\delta(P)$ of a polyhedron P is the smallest integer k such that every pair of vertices of P can be connected by a path using at most k edges of P. The diameter is a fundamental feature of a polyhedron and is closely related to the theoretical complexity of the simplex algorithm; the number of pivots needed, in the worst case, by the simplex algorithm to solve a linear programming problem on a polyhedron P is bounded from below by $\delta(P)$.

One of the outstanding open problems in the areas of polyhedral combinatorics and operations research is to understand the behavior of $\Delta(d, n)$, the maximum possible diameter of a d-dimensional polyhedron with n facets. In 1957, Warren M. Hirsch asked whether $\Delta(d, n) \leq n - d$. While this inequality was shown to hold for $d \leq 3$ [13–15], Klee and Walkup [16] disproved it for unbounded polyhedra when $d \geq 4$ in 1967, and Santos [25] finally disproved it for bounded polyhedra, i.e., for polytopes, in 2012. Santos' lower bound, later refined by Matschke, Santos, and Weibel [20], however, violates $n - d$ by only 5 percent. For the history of the Hirsch conjecture, see [26].

The first subexponential upper bound on $\Delta(d, n)$ is due to Kalai and Kleitman [11] who proved in 1992 that $\Delta(d, n)$ is at most $n^{2+\log_2 d}$. The key ingredient for their proof is a recursive inequality on $\Delta(d, n)$, which we call the *Kalai–Kleitman inequality*. The Kalai–Kleitman inequality was later extended to more general settings such as connected layer families by Eisenbrand et al. [8], and subset partition graphs by Gallagher and Kim [9]. For the corresponding lower bounds, we refer to [8,12].

Refining Kalai and Kleitman's approach, Todd [28] showed in 2014 that $\Delta(d, n) \leq (n - d)^{\log_2 d}$ for $n \geq d \geq 1$. The Todd bound is tight for $d \leq 2$ and coincides with the true value $\Delta(d, d)$, i.e., 0, when $n = d$. Sukegawa and Kitahara [27] slightly improved the Todd bound to $(n - d)^{\log_2 (d-1)}$ for $n \geq d \geq 3$. This upper bound is no longer valid for $d \leq 2$, however, coincides with the Hirsch bound of $n - d$, and is tight for $d = 3$. Gallagher and Kim [10] proved that the same bound holds for the diameter of normal simplicial complexes, and, on the other hand, improved it for polytopes.

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1.1. Main results

In this paper, we improve the Todd bound in high dimensions as follows:

Theorem 1.

(a) $\Delta(d,n) \le (n-d)^{\log_2(d/2)} = (n-d)^{-1+\log_2 d}$ *for* $n \ge d \ge 7$,
(b) $\Delta(d,n) \le (n-d)^{\log_2(d/4)} = (n-d)^{-2+\log_2 d}$ *for* $n \ge d \ge 37$, *and*
(c) $\Delta(d,n) \le (n-d)^{\log_2(16+d/8)} = (n-d)^{-3+\log_2 d+\mathcal{O}(1/d)}$ *for* $n \ge d \ge 1$.

Inequalities (a) and (b) hold for, respectively, $d \ge 7$ and $d \ge 37$, and improve the Todd bound by, respectively, one and two orders of magnitude. Inequality (c) holds for any d, and improves the Todd bound for $d \ge 19$. Note that $\log_2\left(16+\frac{d}{8}\right) = \log_2(d) - 3 + O\left(\frac{1}{d}\right)$ since $\log_e(1+x) \le x$ for $x \ge 0$. Thus, Inequality (c) improves the Todd bound by roughly three orders of magnitude for sufficiently large d.

1.2. Our approach

As in [11,27,28], each inequality stated in Theorem 1 will be proved via an induction on d based on the Kalai–Kleitman inequality. In contrast to [11,27,28], we introduce a way of strengthening Todd's analysis for the inductive step in high dimensions. In this approach, however, we need to check a large number of pairs (d,n) for the base case. To address this issue, we devise a computer-assisted method which is based on the two previously known upper bounds on $\Delta(d,n)$:

(i) $\tilde{\Delta}(d,n)$, an *implicit* upper bound on $\Delta(d,n)$ computed recursively from the Kalai–Kleitman inequality,
(ii) the generalized Larman bound implying $\Delta(d,n) \le 2^{d-3}n$.

The Larman bound of $2^{d-3}n$ was originally proved for bounded polyhedra [19], and improved to $\frac{2n}{3}2^{d-3}$ by Barnette [1]. Considering a more generalized setting, Eisenbrand et al. [8] proved a bound of $2^{d-1}n$ in 2010, before Labbé, Manneville, and Santos [18] established in 2015 an upper bound on the diameter of simplicial complexes implying $\Delta(d,n) \le 2^{d-3}n$.

1.3. Related work

It should be noted that although this paper deals with only the two basic parameters d and n, i.e., the dimension and the number of facets of a polyhedron, there have been studies on other parameters.

A well-known example is the maximum integer coordinate of *lattice polytopes*. In [17], Kleinschmidt and Onn proved that the diameter of a lattice polytope whose vertices are drawn from $\{0,1,\dots,k\}^d$ is at most kd. This is an extension of Naddef [24] showing that the diameter of a 0-1 polytope is at most d. In 2015, Del Pia and Michini [4] improved the Kleinschmidt–Onn bound to $kd - \lceil\frac{d}{2}\rceil$ for $k \ge 2$ and showed that it is tight for $k = 2$, before Deza and Pournin [6] further improved the bound to $kd - \lceil\frac{2d}{3}\rceil - (k-3)$ for $k \ge 3$. On the other hand, considering Minkowski sums of primitive lattice vectors, in [5], Deza, Manoussakis, and Onn provided a lower bound of $\lfloor\frac{(k+1)d}{2}\rfloor$ for $k < d$.

Another well-studied parameter would be Δ_A which is defined as the largest absolute value of a subdeterminant of the constraint matrix A associated to a polyhedron. Bonifas et al. [2] strengthened and extended the Dyer and Frieze upper bound [7] holding for totally unimodular case; i.e., when $\Delta_A = 1$. Complexity analyses based on Δ_A for the shadow vertex algorithm and the primal-simplex based Tardos' algorithm were proposed by Dadush and Hähnle [3], and Mizuno, Sukegawa, and Deza [22,23], respectively.

We also note that there are studies that attempt to understand the behavior of $\Delta(d,n)$ when the number of facets is sufficiently large. Gallagher and Kim [10] provided tail-polynomial upper bounds on the diameter of a normal simplicial complex; specifically, they showed that the diameter is bounded from above by a polynomial in n when n is sufficiently large. An alternative simpler proof for such tail-polynomial upper bounds can be found in Mizuno and Sukegawa [21]. In contrast, by assuming that d is large, rather than n, we strengthen the previous analyses to yield the improved upper bounds.

2. Preliminaries

A *polyhedron* $P \subseteq \mathbb{R}^d$ is an intersection of a finite number of closed halfspaces, and $\dim(P)$ denotes the dimension of the affine hull of P. For a polyhedron P, an inequality $a^\top x \le \beta$ is said to be *valid* for P if it is satisfied by every $x \in P$. We say that F is a *face* of P if there is a valid inequality $a^\top x \le \beta$ for P which satisfies $F = P \cap \{x \in \mathbb{R}^d : a^\top x = \beta\}$. In particular, 0-, 1-, and $(\dim(P)-1)$-dimensional faces are, respectively, referred to as *vertices*, *edges*, and *facets*.

The *diameter* $\delta(P)$ of a polyhedron P is the smallest integer k such that every pair of vertices of P can be connected by a path using at most k edges of P. In this paper, we are concerned with upper bounds on $\Delta(d,n)$, the maximum possible diameter of a d-dimensional polyhedron with n facets. Lemma 1 states the *Kalai–Kleitman inequality* on which our approach is based.

Lemma 1 (*Kalai–Kleitman Inequality [11]*)**.** *For* $\lfloor\frac{n}{2}\rfloor \ge d \ge 2$,

$$\Delta(d,n) \le \Delta(d-1,n-1) + 2\Delta\left(d, \left\lfloor\frac{n}{2}\right\rfloor\right) + 2.$$