Discrete Mathematics 340 (2017) 2143–2160

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Edge-colorings of graphs avoiding complete graphs with a prescribed coloring

Fabrício S. Benevides [a,*], Carlos Hoppen [b], Rudini M. Sampaio [a]

[a] *Universidade Federal do Ceará – Campus do Pici, Bloco 914, 60451–760 Fortaleza, Brazil*
[b] *Instituto de Matemática e Estatística, Universidade Federal do Rio Grande do Sul – Avenida Bento Gonçalves, 9500, 91509-900, Porto Alegre, Brazil*

ARTICLE INFO

Article history:
Received 25 July 2016
Received in revised form 10 April 2017
Accepted 11 April 2017
Available online 15 May 2017

Keywords:
Edge-coloring
Extremal graph
Symmetrization
Hölder
Regularity

ABSTRACT

Given a graph F and an integer $r \geq 2$, a partition $\widehat{F}$ of the edge set of F into at most r classes, and a graph G, define $c_{r,\widehat{F}}(G)$ as the number of r-colorings of the edges of G that do not contain a copy of F such that the edge partition induced by the coloring is isomorphic to the one of F. We think of $\widehat{F}$ as the pattern of coloring that should be avoided. The main question is, for a large enough n, to find the (extremal) graph G on n vertices which maximizes $c_{r,\widehat{F}}(G)$. This problem generalizes a question of Erdős and Rothschild, who originally asked about the number of colorings not containing a monochromatic clique (which is equivalent to the case where F is a clique and the partition $\widehat{F}$ contains a single class). We use Hölder's Inequality together with Zykov's Symmetrization to prove that, for any $r \geq 2$, $k \geq 3$ and any pattern $\widehat{K_k}$ of the clique K_k, there exists a complete multipartite graph that is extremal. Furthermore, if the pattern $\widehat{K_k}$ has at least two classes, with the possible exception of two very small patterns (on three or four vertices), every extremal graph must be a complete multipartite graph. In the case that $r = 3$ and $\widehat{F}$ is a rainbow triangle (that is, where $F = K_3$ and each part is a singleton), we show that an extremal graph must be an almost complete graph. Still for $r = 3$, we extend a result about monochromatic patterns of Alon, Balogh, Keevash and Sudakov to some patterns that use two of the three colors, finding the exact extremal graph. For the later two results, we use the Regularity and Stability Method.

1. Introduction

For any fixed graph F, we say that a graph G is *F-free* if it does not contain F as a subgraph. Finding the maximum number of edges among all F-free n-vertex graphs, and determining the class of n-vertex graphs that achieve this number is known as the *Turán problem* associated with F, which was solved for complete graphs in [21]. The maximum number of edges in an F-free n-vertex graph is denoted by $\mathrm{ex}(n, F)$ and the n-vertex graphs that achieve this bound are called *F-extremal*. Turán has found the value of $\mathrm{ex}(n, F)$ for the case where F is a clique K_k on k vertices, for any $k \geq 3$. Moreover, he showed that the K_k-free graph on n vertices which has $\mathrm{ex}(n, K_k)$ edges is unique (up to isomorphism). This graph is a complete multipartite graph with $k - 1$ parts of sizes as equal as possible, and we will denote it by $T_{k-1}(n)$. This problem and its many variants have been widely studied and there is a vast literature related with it. For more information, see Füredi and Simonovits [5] and the references therein.

* Corresponding author.
E-mail addresses: fabricio@mat.ufc.br (F.S. Benevides), choppen@ufrgs.br (C. Hoppen), rudini@lia.ufc.br (R.M. Sampaio).

http://dx.doi.org/10.1016/j.disc.2017.04.011
0012-365X/

In connection with a question of Erdős and Rothschild [4], several authors have investigated the following related problem. Instead of looking for F-free n-vertex graphs, they were interested in *edge-colorings* of graphs on n vertices such that *every color class is F-free*. (We observe that edge colorings in this work are not necessarily proper colorings.) More precisely, given an integer $r \geq 1$ and a graph F containing at least one edge, one considers the function $c_{r,F}(G)$ that associates, with the graph G, the number of r-colorings of the edge set of G for which there is no monochromatic copy of F. Similarly as before, the problem consists of finding $c_{r,F}(n)$, the maximum of $c_{r,F}(G)$ over all n-vertex graphs G.

The function $c_{r,F}(n)$ has been studied for several classes of graphs, such as complete graphs [1,17,23], odd cycles [1], matchings [7], paths and stars [9]. The hypergraph analogue of this problem has also been considered, see for instance [8,10,14,15], and there has been recent progress in the context of additive combinatorics [6]. There is a straightforward connection between $c_{r,F}(n)$ and $\mathrm{ex}(n, F)$, namely

$$c_{r,F}(n) \geq r^{\mathrm{ex}(n,F)} \text{ for every } n \geq 2, \tag{1}$$

as any r-coloring of the edges of an F-extremal n-vertex graph is trivially F-free, and there are precisely $r^{\mathrm{ex}(n,F)}$ such colorings. For $r \in \{2, 3\}$ the inequality (1) is actually an equation for several graph classes, such as complete graphs [1,23], odd cycles [1] and matchings [7]. On the other hand, for $r \geq 4$ and all connected F, one may easily show that $c_{r,F}(n) > r^{\mathrm{ex}(n,F)}$ (see [1] for non-bipartite graphs and [9, Proposition 3.4] for bipartite graphs).

Here we consider a natural generalization of the above, which was first studied by Lefmann and one of the current authors [11]. Given a k-vertex graph F and a colored graph Γ obtained by coloring the edges of F with at most r colors, we consider the number of r-edge-colorings of a larger graph G that avoids the *color pattern* of Γ. Here, a pattern $\widehat{F}$ of a graph F is defined as any partition of the edge set of F, and the pattern given by a coloring Γ is simply the pattern induced by the color classes. Notice that in a pattern we ignore the name of the colors. We let $c_{r,\widehat{F}}(G)$ denote the number or r-colorings of G which contain no k-vertex subgraph whose color pattern is isomorphic to the one of $\widehat{F}$; naturally, the quantity $c_{r,\widehat{F}}(n)$ is the maximum of $c_{r,\widehat{F}}(G)$ over all n-vertex graphs. We say that a coloring that avoids the pattern of $\widehat{F}$ is $\widehat{F}$-free. When the context is clear we omit the subscripts in $c_{r,\widehat{F}}(G)$ and also refer to an $\widehat{F}$-free r-coloring simply as a *good coloring*. Also, a graph G on n vertices is called $(r, \widehat{F})$-extremal (or simply extremal), when $c_{r,\hat{F}}(G) = c_{r,\hat{F}}(n)$.

We note that Balogh [2] had also considered a multicolored variant of the original Erdős–Rothschild problem. Given F, Γ and G as before, he considered the number $C_{r,\Gamma}(G)$ of r-colorings of G which do not contain a copy of F colored *exactly* as Γ (that is, in his version, we were not allowed to permute the colors). Observe that, if $\widehat{F}$ is the pattern given by Γ $c_{r,\widehat{F}}(G) \leq C_{r,\Gamma}(G)$, but the notions of these two quantities are different. For example, consider the case where Γ is a coloring of F that uses only one of the r colors, say "blue". In this case, $c_{r,\widehat{F}}(G)$ counts the number of colorings of G that avoids monochromatic copies of F, agreeing with the previous definition of $c_{r,F}(G)$, while $C_{r,\Gamma}(G)$ is the number of colorings of G which does not contain a blue copy of F (but may contain monochromatic copies of F in other colors). As another example, if one considers r-colorings of G, but the coloring of Γ uses at most $r-1$ of the colors, then the complete graph K_n is always extremal for $C_{r,\Gamma}(n)$, as the missing color may be used for any edge and hence may be used to extend colorings of any n-vertex graph G to colorings of K_n. However, colorings may not always be extended in this way in the case where we want to avoid color patterns, that is, when we are searching for the extremal graphs of $c_{r,\widehat{F}}(n)$.

Balogh [2] proved that in the case where $r = 2$ and Γ is a 2-coloring of a clique K_k that uses both colors then $C_{2,\Gamma}(n) = 2^{\mathrm{ex}(n,K_k)}$ for n large enough, so the Turán graph $T_{k-1}(n)$ allows the maximum number of 2-colorings with no copy of Γ. (Note that this implies $c_{2,\widehat{F}}(n) = 2^{\mathrm{ex}(n,F)}$ for any pattern of K_k with two classes.) However, the picture changes if we consider 3-colorings with no rainbow triangles (pattern R_0 in Fig. 1): Balogh also observed that, if we color the complete graph K_n with any two of the three colors available, there is no rainbow copy of K_3, which gives at least $3 \cdot 2^{\binom{n}{2}} - 3 \gg 3^{\mathrm{ex}(n,K_3)} = 3^{n^2/4+o(n^2)}$ distinct colorings avoiding rainbow triangles. (As usual, we say that two positive functions g, f satisfy $g(n) \ll f(n)$ if $\lim_{n\to\infty} g(n)/f(n) = 0$.)

In this paper, we focus on the case where $r \geq 3$ and the pattern is given by any edge-coloring of a *clique* that is not monochromatic. The paper has two parts which use very different techniques. In the first part, corresponding to Section 2, we shall use some ideas from the so called Zykov's symmetrization [24] (which also yields one of the classical proofs of Turán's theorem), together with Hölder's Inequality for a certain vector space, to prove a general result that works for arbitrary patterns (including the monochromatic one). First we show the following:

Theorem 1.1. *Let $\widehat{F_k}$ be any r-coloring of K_k. For every natural n, there exists a complete multipartite graph on n vertices which is $(r, \widehat{F_k})$-extremal.*

Very recently, Pikhurko, Staden and Yilma [16] have obtained a similar result, albeit for a different extension of the original problem about monochromatic patterns (their forbidden patterns are still only monochromatic cliques, but they forbid cliques of different sizes for different colors).

In addition, we also proved that whenever the pattern is non-monochromatic and is different than two particular small patterns, then *every* extremal graph is a complete multipartite one.

Theorem 1.2. *Let $r \geq 2$ and $k \geq 3$ be given and let $\widehat{F_k}$ be an r-coloring of K_k which is not monochromatic and is different from the pattern T_0. Also assume that if $r = 2$ then $\widehat{F_k}$ is different from the pattern P_2 (see Fig. 1). Then* every *$(r, \widehat{F_k})$-extremal graph is a complete multipartite graph.*