Discrete Mathematics 340 (2017) 2183–2193

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Poisson approximation of counts of induced subgraphs in random intersection graphs

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ARTICLE INFO

Article history:
Received 5 December 2015
Received in revised form 12 December 2016
Accepted 13 April 2017
Available online 16 May 2017

Keywords:
Stein's method
Poisson approximation
Random intersection graphs

ABSTRACT

Random intersection graphs are characterised by three parameters: n, m and p, where n is the number of vertices, m is the number of objects, and p is the probability that a given object is associated with a given vertex. Two vertices in a random intersection graph are adjacent if and only if they have an associated object in common. When $m = \lfloor n^\alpha \rfloor$ for constant α, we provide a condition, called *strictly α-balanced*, for the Poisson convergence of the number of induced copies of a fixed subgraph.

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1. Introduction

The random intersection graph $\mathcal{G}(n, m, p)$ is a probability distribution on labelled graphs. The set of vertices of the random intersection graph $\mathcal{V}$ is of size $|\mathcal{V}| = n$ and a second set $\mathcal{W}$ of size $|\mathcal{W}| = m$, called the set of *objects*, is used to determine the adjacencies in the graph. Each vertex $v \in \mathcal{V}$ is associated with a set of objects $\mathcal{W}_v \subseteq \mathcal{W}$ and two vertices $v_1, v_2 \in \mathcal{V}$ are adjacent if and only if $\mathcal{W}_{v_1} \cap \mathcal{W}_{v_2} \neq \emptyset$. The randomness in the graph comes by setting $\mathbb{P}\{w \in \mathcal{W}_v\} = p$ independently for all $w \in \mathcal{W}$, $v \in \mathcal{V}$. The preceding description characterises the random intersection graph denoted by $\mathcal{G}(n, m, p)$. The model $\mathcal{G}(n, m, p)$ was introduced in [6].

Let H_0 be a given graph on $h \geq 2$ vertices and with at least one edge and let $K_{\mathcal{V}}$ denote the complete graph on the vertex set $\mathcal{V}$. Let $\mathcal{H}_0$ denote the set of subgraphs of $K_{\mathcal{V}}$ isomorphic to H_0. A copy $H \in \mathcal{H}_0$ is *induced* in $\mathcal{G}(n, m, p)$ if all of its edges are edges in $\mathcal{G}(n, m, p)$ and none of its non-edges are edges in $\mathcal{G}(n, m, p)$. In this paper we find conditions on H_0, n, m and p which imply that the number of induced copies of H_0 in $\mathcal{G}(n, m, p)$ has an approximately Poisson distribution.

We denote the number of induced copies of H_0 in $\mathcal{G}(n, m, p)$ by $X = X(H_0)$. In order to facilitate our Poisson approximation of the distribution of X, we will express X as a sum of indicator random variables. Given an integer N, define the set $[N]$ to be $[N] = \{1, \ldots, N\}$ and define $\mathrm{aut}(H_0)$ to be the set of automorphisms of H_0. The number of subgraphs of $K_{\mathcal{V}}$ isomorphic to H_0 is

$$N_n = |\mathcal{H}_0| = \binom{n}{h} \frac{h!}{|\mathrm{aut}(H_0)|}$$

and we may index the subgraphs in $\mathcal{H}_0$ by

$$\mathcal{H}_0 = \{H_i : i \in [N_n]\}.$$

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http://dx.doi.org/10.1016/j.disc.2017.04.013
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We decompose X as

$$X = \sum_{i \in [N_n]} X_i, \tag{1}$$

where X_i is the indicator random variable of the event $\{H_i$ is induced in $\mathcal{G}(n, m, p)\}$. The intention is that the X_i's should be approximately independent and therefore X should approach a Poisson distribution as $n \to \infty$ for appropriate choices of m and p.

The total variation distance between a random variable taking nonnegative integer values and a random variable P_λ with the Poisson distribution with parameter λ is defined to be

$$d_{TV}(X, P_\lambda) = \frac{1}{2} \sum_{k=0}^{\infty} \left| \mathbb{P}\{X = k\} - e^{-\lambda}\lambda^k/k! \right|.$$

As was done in [6,8], we parametrise $m = m(n)$ by

$$m = \lfloor n^\alpha \rfloor \tag{2}$$

for some constant $\alpha > 0$. Our method of proof will be to apply Stein's method to show that $d_{TV}(X, P_\lambda) \to 0$ as $n \to \infty$ under suitable conditions.

Poisson approximation for the number of induced copies of subgraphs has already been studied in detail for the Erdős–Rényi model of random graphs, in which edges appear independently and with identical probability $\hat{p}$ (see Chapter 6 of [4]). Let H_0 be a graph with e edges and h vertices. Let $V(H_0)$ and $E(H_0)$ be the vertex and edge sets of H_0, respectively. Given $S \subseteq V(H_0)$, we define $E_S(H_0)$ to be the set of edges of H_0 having both vertices in S. A graph H_0 is called *strictly balanced* if

$$\max_{\emptyset \subsetneq S \subsetneq V(H_0)} \frac{|E_S(H_0)|}{|S|} < \frac{e}{h}. \tag{3}$$

Let $W = W(H_0)$ denote the number of not necessarily induced copies of H_0 in $\mathcal{G}\left(n, \hat{p}\right)$ and let

$$\lambda = \lambda(H_0) = \mathbb{E}(W(H_0)) = \binom{n}{h} \frac{h!}{|\mathrm{aut}(H_0)|} \hat{p}^e.$$

Define $\kappa = \kappa(H_0)$ by

$$\kappa = \min_{\emptyset \subsetneq S \subsetneq V(H_0)} |E_S(H_0)| \left(\frac{|S|}{|E_S(H_0)|} - \frac{h}{e} \right).$$

Bollobás [2] shows Poisson convergence of W through the method of moments. Theorem 5.B of [1] gives the bound

$$d_{\mathrm{TV}}(W, P_\lambda) = \begin{cases} O(1)\lambda^{1-1/e} n^{-\kappa} & \text{if } \lambda \geq 1; \\ O(1)\lambda\, n^{-\kappa} & \text{if } \lambda < 1. \end{cases} \tag{4}$$

When $\hat{p}$ is such that $\lambda \to \lambda_0$ for a constant λ_0, then (4) implies that the distribution of W converges in total variation distance to a Poisson Po (λ_0) distribution. That is not the case for subgraphs which are not strictly balanced.

The only subgraphs for which the asymptotic distribution of $X(H_0)$ has been determined for $\mathcal{G}(n, m, p)$ are $H_0 = K_h$, the complete graphs on h vertices, in [8], in which $X(K_h)$ was shown to have a limiting Poisson distribution at the threshold for the appearance of K_h. Theorem 1 from [8] for complete graphs is the kind of result we have in mind to extend to general H_0. For a constant $c > 0$, we parametrise $p = p(n)$ by

$$p(n) \sim \begin{cases} c\, n^{-1} m^{-\frac{1}{h}} & \text{for } 0 < \alpha < \dfrac{2h}{h-1}; \\ c\, n^{-\frac{h+1}{h-1}} & \text{for } \alpha = \dfrac{2h}{h-1}; \\ c\, n^{-\frac{1}{h-1}} m^{-\frac{1}{2}} & \text{for } \alpha > \dfrac{2h}{h-1}. \end{cases} \tag{5}$$

We focus on asymptotic values thus we will use standard Landau notation $O(\cdot)$, $o(\cdot)$, $\Omega(\cdot)$, $\sim$, and $\asymp$ as in [4]. The following theorem was proved in [8].

Theorem 1. *Let $\mathcal{G}(n, m, p)$ be a random intersection graph defined with m and p given in terms of n by* (2) and (5) *and let $h \geq 3$ be a fixed integer. Let $X(K_h)$ be the random variable counting the number of instances of K_h in $\mathcal{G}(n, m, p)$.*

(i) *If $\alpha < \frac{2h}{h-1}$, then $\lambda_n = \mathbb{E}X(K_h) \sim c^h/h!$ and*

$$d_{TV}(X(K_h), P_{\lambda_n}) = O\left(n^{-\frac{\alpha}{h}}\right);$$

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