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Forbidden pairs of disconnected graphs for traceability in connected graphs

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ABSTRACT

Let $\mathcal{H}$ be a class of given graphs. A graph G is said to be $\mathcal{H}$-free if G contains no induced copies of H for every $H \in \mathcal{H}$. A graph is called traceable if it contains a Hamilton path. Faudree and Gould (1997) characterized all the pairs $\{R, S\}$ of connected subgraphs such that every connected $\{R, S\}$-free graph is traceable. Li and Vrána (2016) first consider the disconnected forbidden subgraphs, and characterized all pairs of disconnected forbidden subgraphs for hamiltonicity. In this article, we characterize all pairs $\{R, S\}$ of graphs such that there exists an integer n_0 such that every connected $\{R, S\}$-free graph of order at least n_0 is traceable.

1. Introduction

We use Bondy and Murty [2] for terminology and notations not defined here and consider finite simple graphs only.

Let $G = (V(G), E(G))$ be a graph. We use $n(G)$, $e(G)$, $\alpha(G)$, $\kappa(G)$ and $\omega(G)$ to denote the *order, size, independence number, connectivity* and *component number* of G, respectively. Let u be a vertex of G. We use $N_G(u)$ to denote the set of vertices which is adjacent with u (also called the *neighbors* of u) in the graph G. Let S be a subset of $V(G)$(or $E(G)$). The *induced subgraph* of G is denoted by $G[S]$. Furthermore, we use $G - S$ to denote the subgraph $G[V(G)\backslash S]$.

Let H be a given graph. A graph G is said to be *H-free* if G contains no induced copies of H. If G is H-free, then H is called a *forbidden subgraph* of G. For a class of graphs $\mathcal{H}$, the graph G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. Note that if H_1 is an induced subgraph of H_2, then every H_1-free graph is also H_2-free.

As usual, we use K_n to denote the complete graph of order n, and $K_{m,n}$ to denote the complete bipartite graph with partition sets of size m and n. So the K_1 is a vertex, K_3 is a triangle, $K_{1,r}$ is a star (the $K_{1,3}$ is also called a claw). The clique C is a subset of vertices of a graph G such that $G[C]$ is a complete graph. Then we will show some special graphs which are needed: (see Fig. 1)

- P_i, the path with i vertices (note that $P_1 = K_1$ and $P_2 = K_2$);
- Z_i, a graph obtained by identifying a vertex of a K_3 with an end-vertex of a P_{i+1};
- L_i, a graph obtained by identifying a vertex of a K_i with an end-vertex of a P_2;
- $N_{i,j,k}$, a graph obtained by identifying the three vertices of a K_3 with an end-vertex of a P_{i+1}, P_{j+1} and P_{k+1}, respectively.

A graph is called *hamiltonian* if it contains a Hamilton cycle, and is called *traceable* if it contains a Hamilton path. Bedrossian [1] characterized all pairs of connected forbidden subgraphs for hamiltonicity. Faudree and Gould [6] extended

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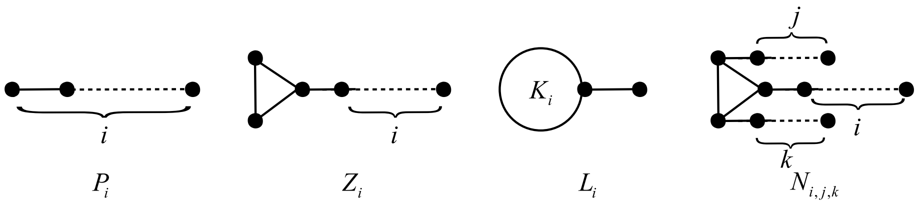

Fig. 1. Some special graphs: P_i, Z_i, L_i and $N_{i,j,k}$.

Bedrossian's result by proving the necessity part of the theorem based on infinite families of non-hamiltonian graphs, and characterized all the pairs $\{R, S\}$ of connected subgraphs such that every connected $\{R, S\}$-free graph is traceable.

Theorem 1 ([6]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,1}$.*

In [7], Li and Vrána first consider disconnected forbidden subgraphs, and characterized all pairs of disconnected forbidden subgraphs for hamiltonicity.

Theorem 2 ([7]). *Let S be a graph of order at least three. Then every 2-connected S-free graph is hamiltonian if and only if S is P_3 or $3K_1$.*

Theorem 3 ([7]). *Let R and S be graphs of order at least three other than P_3 and $3K_1$. Then there is an integer n_0 such that every 2-connected $\{R, S\}$-free graph of order at least n_0 is hamiltonian, if and only if, one of the following is true (up to symmetry):*

- *$R = K_{1,3}$ and S is an induced subgraph of $P_6, Z_3, N_{0,1,2}, N_{1,1,1}, K_1 \bigcup Z_2, K_2 \bigcup Z_1$, or $K_3 \bigcup P_4$;*
- *$R = K_{1,k}$ with $k \geq 4$ and S is an induced subgraph of $2K_1 \bigcup K_2$;*
- *$R = kK_1$ with $k \geq 4$ and S is an induced subgraph of L_l with $l \geq 3$, or $2K_1 \bigcup K_{l'}$ with $l' \geq 2$.*

In this paper, we also consider disconnected forbidden subgraphs for traceability. First, we characterize all graphs S such that every connected S-free graph is traceable.

Theorem 4. *Let S be a graph of order at least three. Then every connected S-free graph is traceable if and only if S is P_3 or $3K_1$.*

Then we will characterize all pairs of disconnected $\{R, S\}$ of graphs such that there exists an integer n_0 such that every connected $\{R, S\}$-free graph of order at least n_0 is traceable by the following theorems.

Theorem 5. *Let G be a connected $K_{1,3}$-free graph. Then*

(1) *if G is also $K_1 \bigcup Z_1$-free and $n(G) \geq 13$, then G is traceable;*

(2) *if G is also $2K_1 \bigcup K_3$-free and $n(G) \geq 31$, then G is traceable.*

Theorem 6. *Let G be a connected $\{K_{1,k}, 2K_1 \bigcup K_2\}$-free graph, where $k \geq 4$. If $n(G) \geq 2k + 1$, then G is traceable.*

The Ramsey number $r(k, l)$ is defined as the smallest integer such that every graph of order $r(k, l)$ contains either a clique of k vertices or an independent set of l vertices [9].

Theorem 7. *Let G be a connected kK_1-free graph, where $k \geq 4$. Then*

(1) *if G is also L_l-free, where $l \geq 3$, and $n(G) \geq r(2l - 3, k) + k - 2$, then G is traceable;*
(2) *if G is also $2K_1 \bigcup K_{l'}$-free, where $l' \geq 2$, and $n(G) \geq r(2k + l' - 2, k)$, then G is traceable.*

Finally, we may state our main result as follows.

Theorem 8. *Let R and S be graphs of order at least three other than P_3 and $3K_1$. Then there is an integer n_0 such that every connected $\{R, S\}$-free graph of order at least n_0 is traceable, if and only if, one of the following is true (up to symmetry):*

- *$R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,1}, K_1 \bigcup Z_1$, or $2K_1 \bigcup K_3$;*
- *$R = K_{1,k}$ with $k \geq 4$ and S is an induced subgraph of $2K_1 \bigcup K_2$;*
- *$R = kK_1$ with $k \geq 4$ and S is an induced subgraph of L_l with $l \geq 3$, or $2K_1 \bigcup K_{l'}$ with $l' \geq 2$.*

Note that there is a slight difference between Theorems 3 and 8 only in the first item.