Discrete Mathematics 340 (2017) 2208–2216

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Bounded clique cover of some sparse graphs

Andrea Munaro

Laboratoire G-SCOP, Univ. Grenoble Alpes, France

ARTICLE INFO

Article history:
Received 9 June 2016
Received in revised form 6 December 2016
Accepted 3 April 2017
Available online 21 June 2017

Keywords:
Clique cover
θ-bounded
Bounded degree graphs
PTAS

ABSTRACT

We show that $f(x) = \lfloor \frac{3}{2}x \rfloor$ is a θ-bounding function for the class of subcubic graphs and that it is best possible. This generalizes a result by Henning et al. (2012), who showed that $\theta(G) \leq \frac{3}{2}\alpha(G)$ for any subcubic triangle-free graph G. Moreover, we provide a θ-bounding function for the class of K_4-free graphs with maximum degree at most 4. Finally, we study the problem Clique Cover for subclasses of planar graphs and graphs with bounded maximum degree: in particular, answering a question of Cerioli et al. (2008), we show it admits a PTAS for planar graphs.

1. Introduction

A *clique* of a graph is a set of pairwise adjacent vertices, a *clique cover* is a set of cliques such that each vertex of the graph belongs to at least one of them and an *independent set* is a set of pairwise non-adjacent vertices. We denote by $\theta(G)$ and $\alpha(G)$ the minimum size of a clique cover and the maximum size of an independent set of the graph G, respectively. Clearly, for any graph G, we have $\theta(G) \geq \alpha(G)$ and a class of graphs $\mathcal{G}$ is *θ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for all $G \in \mathcal{G}$ and all induced subgraphs H of G, we have $\theta(H) \leq f(\alpha(H))$. Such a function f is a *θ-bounding* function for $\mathcal{G}$. The notion of θ-boundedness and its complementary χ-boundedness were introduced by Gyárfás [11] in order to provide a natural extension of the class of perfect graphs: indeed, this class is exactly the class of graphs θ-bounded by the identity function. One of the main questions formulated in [11] is the following: given a class $\mathcal{G}$, what is the smallest θ-bounding function for $\mathcal{G}$, if any? We answer this question for the class of subcubic graphs:

Theorem 1. *If G is a subcubic graph, then $\theta(G) \leq \frac{3}{2}\alpha(G)$. Moreover, $f(x) = \lfloor \frac{3}{2}x \rfloor$ is the smallest θ-bounding function for the class of subcubic graphs.*

Elaborating on a result by Choudum et al. [7], Pedersen conjectured that $\theta(G) \leq \frac{3}{2}\alpha(G)$, for any subcubic triangle-free graph G (see [15]). Recall that, if G is a triangle-free graph and $\alpha'(G)$ denotes the maximum size of a matching in G, then $\theta(G) = \alpha'(G) + (|V(G)| - 2\alpha'(G)) = |V(G)| - \alpha'(G)$. Pedersen's conjecture was confirmed by Henning et al. [15], who actually proved the following generalization:

Theorem 2 (*Henning et al. [15]*). *If G is a subcubic graph, then*

$$\frac{3}{2}\alpha(G) + \alpha'(G) + \frac{1}{2}t(G) \geq |V(G)|,$$

where $t(G)$ denotes the maximum number of vertex-disjoint triangles of G. Moreover, equality holds if and only if every component of G is in $\{K_3, K_4, C_5, G_{11}\}$ (see Fig. 1).

E-mail address: Andrea.Munaro@grenoble-inp.fr.

http://dx.doi.org/10.1016/j.disc.2017.04.004
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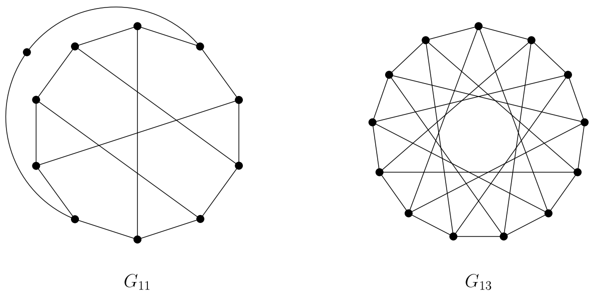

Fig. 1. The graphs G_{11} and G_{13}.

Theorem 2 implies that $f(x) = \lfloor \frac{3}{2}x \rfloor$ is the smallest θ-bounding function for the class of subcubic triangle-free graphs. Consider now the class $\mathcal{C}$ containing those graphs G such that $\alpha(H) \geq \frac{|V(H)|}{3}$, for every induced subgraph H of G. Gyárfás et al. [12] showed that $f(x) = \lfloor \frac{8}{5}x \rfloor$ is the smallest θ-bounding function for the class $\mathcal{C}$. In particular, they proved the following:

Theorem 3 (*Gyárfás et al. [12]*)**.** *If $G \in \mathcal{C}$, then $\theta(G) \leq \frac{8}{5}\alpha(G)$.*

By Brooks' Theorem [21], every connected subcubic graph different from K_4 belongs to $\mathcal{C}$ and so $f(x) = \lfloor \frac{8}{5}x \rfloor$ is a θ-bounding function for the class of subcubic graphs as well. On the other hand, Gyárfás et al. [13] provided evidence for the following meta-statement: the graphs for which the difference $\theta - \alpha$ is large are triangle-free. It would therefore be natural to expect that the ratio $\frac{\theta}{\alpha}$ is maximum for triangle-free graphs and Theorem 1 partially confirms this intuition.

Our proof of Theorem 1 in Section 2 is inspired by that of Theorem 3. The main idea is rather simple and it is based on the notion of θ-criticality, a graph G being θ-*critical* if $\theta(G - v) < \theta(G)$, for every $v \in V(G)$. First, we show that a minimum counterexample is connected and θ-critical. We then rely on the following result by Gallai (see [19] for a short proof and an extension):

Theorem 4 (*Gallai [9]*)**.** *If v is any vertex of a connected θ-critical graph G, then G has a minimum-size clique cover in which v is the only isolated vertex. In particular, $\theta(G) \leq \frac{|V(G)|+1}{2}$.*

The final contradiction is then reached by using an appropriate lower bound for the independence number of a subcubic graph.

Let us now consider the class of graphs with maximum degree at most 4. Joos [16] relaxed the degree condition in Theorem 2 and showed that $\theta(G) \leq \frac{7}{4}\alpha(G)$, for any triangle-free graph G with $\Delta(G) \leq 4$:

Theorem 5 (*Joos [16]*)**.** *If G is a triangle-free graph with $\Delta(G) \leq 4$, then $\frac{7}{4}\alpha(G) + \alpha'(G) \geq |V(G)|$. Moreover, equality holds if and only if every component C of G has order 13, $\alpha(C) = 4$ and $\alpha'(C) = 6$.*

It would be tempting to extend Theorem 5 to the class of graphs with maximum degree 4 in the same way we extend Theorem 2 to the class of subcubic graphs. Unfortunately, the method adopted in the proof of Theorem 1 does not seem to be powerful enough for this purpose and the price we have to pay is a bigger θ-bounding function (see Theorem 18), likely to be far from the optimal.

In Section 3, we consider the problem of finding a clique cover of minimum size. The decision version of this well-known NP-complete problem is formulated as follows:

Clique Cover	
Instance:	A graph G and a positive integer k.
Question:	Does $\theta(G) \leq k$ hold?

Since any subset of a clique is again a clique, Clique Cover is equivalent to the following problem:

Clique Partition	
Instance:	A graph G and a positive integer k.
Question:	Does there exist a partition of $V(G)$ into k disjoint cliques?

Moreover, Clique Partition is clearly equivalent to the well-known Colouring problem on the complement graph.